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Par
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Growth in Groups of Non-Positive Curvature

Croissance dans les groupes à courbure négative ou nulle
Crecimiento en grupos de curvatura negativa o nula

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Universidad Complutense de Madrid

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TESIS DOCTORAL

Growth in Groups of Non-positive Curvature

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Croissance dans les groupes à courbure négative ou nulle

Memoria para optar al grado de doctor presentada por

Xabier Legaspi Juanatey

Directores :

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UNIVERSIDAD COMPLUTENSE MADRID

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Growth in Groups of Non-positive Curvature

y dirigida por: Prof. Yago Antolín Pichel y Prof. Rémi Coulon

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quen me ensinou a resolver crebacabezas,
e espertou á miña curiosidade polas palabras.

E ao meu pai,
quen me ensinou a desfrutar da vida,
sen perder nunca de vista os soños.

Friends, Romans, countrymen, lend me your ears;
I come to bury Caesar, not to praise him.
The evil that men do lives after them;
The good is oft interred with their bones;
So let it be with Caesar.

---

from *Julius Caesar* of William Shakespeare,
spoken by Marc Antony

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Resumen en español

A una isla del caribe
He tenido que emigrar
Y trabajar de camarero
Lejos lejos de mi hogar
De mi hogar.

Miña Terra Galega, de Siniestro Total

Esta tesis se centra en preguntas que comparan números fáciles de definir pero no fáciles de calcular. La acción de un grupo G sobre un espacio métrico X se dice *propia* si para cada $r > 0$, y para cada $x \in X$, el número de elementos $u \in G$ que mueven x a distancia a lo sumo r es finito. Sea G un grupo actuando mediante isometrías y propiamente sobre un espacio métrico X. La *tasa de crecimiento exponencial relativa* de la acción de un subconjunto $U \subset G$ sobre X es el número

$$\omega(U, X) = \limsup_{r \to \infty} \frac{1}{r} \log |\{ u \in U \ : \ |ux - x| \leqslant r \}|,$$

cuyo valor es independiente del punto $x \in X$. Si G es el grupo fundamental de una variedad hiperbólica cerrada M que actúa sobre el espacio recubridor universal X, entonces $\omega(G, X)$ tiene numerosas interpretaciones. Coincide con la *entropía de volumen* de la variedad M, [71, 62]; el *exponente crítico* de la serie de Poincaré de G, [67, 76]; la *entropía topológica* del flujo geodésico en el fibrado tangente unitario de M, [60]; la *dimensión de Hausdorff* del conjunto límite radial de G, [28], etc. En este contexto, el número $\omega(G, X)$ es la piedra angular que une grupos, geometría y dinámica. La discreción de la órbita de G y la curvatura negativa de M juegan un papel determinante en este fenómeno.

El objetivo de esta tesis es cuantificar el crecimiento en grupos a partir de sus acciones mediante isometrías sobre espacios métricos. El enfoque consiste en observar desde un punto de vista muy lejano. La hazaña está en la finitud y las condiciones de curvatura negativa o nula de acciones y espacios. Sean δ, κ, $N > 0$. La acción de un grupo G sobre un espacio δ-hiperbólico X se dice (κ, N)-*acilíndrica*, [72, 18, 66, 38], si para cada par de puntos $x, y \in X$ que distan al menos κ, el número de elementos $u \in G$ que mueve cada uno de los puntos x, y a una distancia de a lo sumo 100δ está acotado superiormente por N. La última década ha estado enfocada en el estudio de grupos que admiten una acción acilíndrica sobre un espacio hiperbólico en el sentido de Gromov, [66]. Esta es una familia muy amplia de grupos que incluye grupos relativamente hiperbólicos, grupos de cancelación pequeña clásica infinitamente presentados, grupos modulares de superficies, grupos de automorfismos exteriores de grupos libres, grupos de Artin de ángulo recto, etc. Resulta que la mayoría de las veces los grupos que actúan acilíndricamente sobre un espacio hiperbólico también admiten acciones propias sobre otros espacios que no son necesariamente hiperbólicos, pero que contienen isometrías que se comportan como las isometrías loxodrómicas de un espacio hiperbólico: *elementos constrictor*, [74], bajo la terminología de [8]. De hecho, el recíproco siempre es cierto.

La moraleja de la tesis recoge la siguiente idea de M. Gromov: bajo un punto de vista global curvado de forma negativa o nula, todavía es posible producir resultados sólidos para un grupo típico, lo que a veces puede aproximar nuestra comprensión de los grupos monstruo. Estudiaremos dos problemas diferentes usando argumentos de baja tecnología que involucran la desigualdad triangular. El primero versará sobre el crecimiento de subgrupos cuasi-convexos en grupos actuando propiamente con un elemento constrictor. A mayores, hemos añadido un apéndice en dónde se describen algunas características elementales de la propiedad de constricción. El segundo versará sobre el crecimiento uniforme en cocientes de cancelación pequeña sobre grupos que actúan acilíndricamente sobre un espacio hiperbólico. Los Capítulos 1 y 2 corresponden respectivamente a los siguientes artículos:

- X. Legaspi. Constricting elements and the growth of quasi-convex subgroups, 2022. URL: `https://orcid.org/0000-0002-1497-6448`.
- X. Legaspi and M. Steenbock. Uniform uniform exponential growth in small cancellation groups, 2023. URL: `https://orcid.org/0000-0002-1497-6448`.

Introduction en français

> L'absurdité est surtout le divorce de l'homme et du monde.
>
> *L'étranger*, d'Albert Camus

0.1 Résumé

Cette thèse est centrée au tour des questions qui comparent des nombres faciles à définir mais pas faciles à calculer. L'action d'un groupe G sur un espace métrique X est *propre* si pour tout $r > 0$, et pour tout $x \in X$, le nombre d'éléments $u \in G$ qui déplacent x à distance au plus r est fini. Soit G un groupe agissant par isométries et proprement sur un espace métrique X. Le *taux de croissance exponentiel relatif* de l'action d'un sous-ensemble $U \subset G$ sur X est le nombre

$$\omega(U, X) = \limsup_{r \to \infty} \frac{1}{r} \log |\{\, u \in U \ : \ |ux - x| \leqslant r \,\}|,$$

dont la valeur est indépendante du point $x \in X$. Si G est le groupe fondamental d'une variété hyperbolique fermée M agissant sur le revêtement universel X, alors $\omega(G, X)$ a de nombreuses interprétations. Elle correspond à l'*entropie de volume* de la variété M, [71, 62] ; l'*exposant critique* de la série de Poincaré de G, [67, 76] ; l'*entropie topologique* du flot géodésique dans le fibré unitaire tangent de M, [60] ; la *dimension de Hausdorff* de l'ensemble radial limite de G, [28], etc. Dans ce contexte, le nombre $\omega(G, X)$ est la pierre angulaire qui unit les groupes, la géométrie et la dynamique. L'orbite discrète de G et la courbure négative de M jouent un rôle déterminant dans ce phénomène.

L'objectif de cette thèse est de quantifier la croissance des groupes à partir de leurs actions par isométries sur des espaces métriques. L'approche consiste à observer d'un point de vue très éloigné. L'exploit est dans la finitude et les conditions de courbure négative ou nulle des actions et des espaces. Soient δ, κ, $N > 0$. L'action d'un groupe G sur un espace δ-hyperbolique X est (κ, N)-*acylindrique*, [72, 18, 66, 38], si pour chaque paire de points $x, y \in X$ distants d'au moins κ, le nombre d'éléments $u \in G$ qui déplacent chacun des points x, y d'une distance d'au plus 100δ est borné supérieurement par N. La dernière

décennie a été consacrée à l'étude des groupes qui admettent une action acylindrique sur un espace Gromov hyperbolique, [66]. Il s'agit d'une très large famille de groupes qui comprend des groupes relativement hyperboliques, des groupes de petite simplification à présentation infinie, des groupes modulaires de surfaces, des groupes d'automorphismes extérieurs de groupes libres, des groupes d'Artin à angle droit, etc. Il s'avère que la plupart du temps les groupes qui agissent de manière acylindrique sur un espace hyperbolique admettent aussi des actions propres sur d'autres espaces qui ne sont pas forcément hyperboliques, mais qui contiennent des isométries qui se comportent comme les isométries loxodromiques d'un espace hyperbolique : *éléments constricteurs* , [74], sous la terminologie de [8]. En fait, la réciproque est toujours vrai.

La morale de la thèse reprend l'idée suivante de M. Gromov : sous un point de vue global courbé de façon négative ou nulle, il est encore possible de produire des résultats robustes pour un groupe typique, ce qui peut parfois rapprocher notre compréhension à des groupes monstre. Nous étudierons deux problèmes différents en utilisant des arguments de basse technologie impliquant l'inégalité triangulaire. Le premier traitera de la croissance de sous-groupes quasi-convexes dans les groupes agissant proprement avec un élément constricteur. De plus, nous avons ajouté une annexe décrivant quelques conséquences élémentaires de la propriété de constriction. Le second traitera de la croissance uniforme dans les quotients à petites simplifications sur des groupes agissant de manière acylindrique sur un espace hyperbolique. Les Chapitres 1 et 2 correspondent respectivement aux articles suivants :

- X. Legaspi. Constricting elements and the growth of quasi-convex subgroups, 2022. URL: `https://orcid.org/0000-0002-1497-6448`.
- X. Legaspi and M. Steenbock. Uniform uniform exponential growth in small cancellation groups, 2023. URL: `https://orcid.org/0000-0002-1497-6448`.

0.2 Croissance des sous-groupes quasi-convexes

Il existe une quantité importante d'informations codées dans la géométrie des sous-groupes quasi-convexes d'un groupe. Par exemple, certains groupes hyperboliques bénéficient de la propriété qu'un sous-groupe est quasi-convexe si et seulement s'il est de type fini. Cependant, dans d'autres contextes, c'est loin d'être vrai. Dans le Chapitre 2 nous explorons la croissance de sous-groupes quasi-convexes au-delà du cas hyperbolique.

Nous donnons quelques définitions. Soit G un groupe agissant par isométries sur un espace métrique. Afin de définir des notions très générales de courbure négative ou nulle et de cocompacité convexe, nous utilisons des systèmes de chemins, introduits par A. Sisto dans [74]. Grossièrement, un système de chemins $\mathscr{P}$ de X est une collection appropriée de quasi-géodésiques uniformes joignant chaque paire de points de X. Par exemple, les groupes modulaires des surfaces sont accompagnés de chemins hiérarchiques, une famille de quasi-géodésiques spéciales codant des informations substantielles sur la géométrie de l'espace et plus faciles à utiliser que l'ensemble de toutes les (quasi-)géodésiques. Soit $\mathscr{P}$ un système de chemins de X. Soit $\delta \geqslant 0$. On dit qu'un sous-ensemble Y de X est *δ-constricteur* s'il existe une *projection à large échelle au point le plus proche* de X sur A avec la propriété que tout $\gamma \in \mathscr{P}$ joignant n'importe quelle paire de points $x, y \in X$ dont les projections p et q sont δ-loin passe par les δ-voisinages de p et q. Un *élément* g de G est *δ-constricteur* s'il est d'ordre infini et s'il existe une orbite δ-constrictrice du sous-groupe cyclique engendré par g. Soit $\eta \geqslant 0$. Un sous-ensemble Y de X est *η-quasi-convexe* si tout $\gamma \in \mathscr{P}$ avec des extrémités dans Y est contenu dans la η-voisinage de Y. Un sous-groupe H de G est η-quasi-convexe s'il existe une orbite η-quasi-convexe de H.

Example 0.2.1. — Un espace métrique X est *δ-hyperbolique* s'il est géodésique et si tout segment géodésique de X est δ-constricteur par rapport au système de chemins constitué de tous les segments géodésiques de X, [61]. En particulier, l'axe des isométries loxodromiques des espaces δ-hyperboliques est constricteur : cette propriété est en fait équivalente à la quasi-convexité dans les espaces δ-hyperboliques, [26], mais pas en général. Par exemple, une géodésique dans le plan euclidien.

Example 0.2.2. — Voici des exemples de groupes agissant avec un élément constricteur sur chacun de leurs graphes de Cayley localement finis, voir par exemple [8, 57] et les références qui s'y trouvent.

(i) Groupes relativement hyperboliques.

(ii) Groupes modulaires des surfaces.

(iii) Groupes CAT(0) avec éléments Morse.

(iv) Groupes à petit simplification graphique $Gr'(1/6)$.

Nous mentionnons maintenant deux résultats. Le premier est une généralisation de [4] (voir aussi [47]) et étudie les taux de croissance exponentiels relatifs associés aux graphes de Schreier.

Theorem 0.2.3. — *Soit G un groupe agissant proprement sur un espace métrique X. Soit $\mathscr{P}$ un système de chemins de X. Supposons que G contienne un élément constricteur par rapport à $\mathscr{P}$. Soit H un sous-groupe quasi-convexe d'indice infini de G par rapport à $\mathscr{P}$. Il existe $x_0 \in X$ avec la propriété suivante. Soit G_H un ensemble de représentants de G/H tel que $|gx_0 - x_0| = \inf_{h \in H} |ghx_0 - x_0|$, pour tout $g \in G_H$. Alors $\omega(G_H, X) = \omega(G, X)$.*

Le deuxième résultat est une généralisation de [37] (voir aussi [27]) et étudie les taux de croissance exponentiels relatifs associés aux sous-groupes. On dit que l'action propre des isométries d'un sous-ensemble Λ sur un espace métrique X est *divergente* lorsque la *série de Poincaré* $\mathscr{P}_U(s) = \sum_{u \in U} e^{-s|ux-x|}$ diverge à son exposant critique $\omega(U, X)$. Ce comportement est indépendant de $x \in X$.

Theorem 0.2.4. — *Soit G un groupe agissant proprement sur un espace métrique X. Soit $\mathscr{P}$ un système de chemins de X. Supposons que G contienne un élément constricteur par rapport à $\mathscr{P}$. Soit H un sous-groupe quasi-convexe d'indice infini par rapport à $\mathscr{P}$. Si $\omega(H, X) < \infty$ et l'action de H sur X est divergente, alors $\omega(H, X) < \omega(G, X)$.*

Dans le théorème précédent, il existe de nombreuses situations dans lesquelles l'action de H sur X est divergente. Par exemple, si $\mathscr{P}$ est le système de chemins de X composé de tous les segments géodésiques, alors H est quasi-convexe au sens classique. Dans cette situation, la fonction de croissance relative de H est sous-multiplicative, et par conséquent l'action de H sur X est divergente [39] (à condition que H soit infini). Une autre situation dans laquelle la fonction de croissance relative est sous-multiplicative est lorsque H a la propriété Morse ou lorsque la fonction de croissance relative est une fonction purement exponentielle, sans autre hypothèse sur $\mathscr{P}$. Cela permet d'appliquer le résultat aux groupes modulaires des surfaces de type fini et leurs sous-groupes convexes cocompacts ou à certains stabilisateurs de multicourbes. Ici, le rôle du système de chemins est joué par les chemins hiérarchiques.

0.3 Croissance exponentielle uniforme uniforme

Une question ouverte demande si chaque groupe agissant de manière acylindrique sur un espace hyperbolique a une croissance exponentielle uniforme. Dans le Chapitre 2, on montre que la classe des groupes de croissance exponentielle uniforme uniforme agissant de manière acylindrique sur un espace hyperbolique est fermée en prenant les quotients à petites simplifications géométriques $C''(\lambda, \varepsilon)$ dans le sens de [38, Définition 6.22]. Encore une fois, nous commençons par quelques définitions.

Soit G un groupe. Soit U un sous-ensemble symétrique fini de G, notons H le sous-groupe engendré par U, et soit X_U le graphe de Cayley correspondant. Le *taux de croissance exponentiel de U* est le nombre

$$\omega(H, U) := \omega(H, X_U).$$

Soit $\xi > 0$. On dit que G a *croissance exponentielle ξ-uniforme* s'il est de type fini et pour tout ensemble générateur symétrique fini U de G, on a $\omega(G, U) > \xi$. On dit que G a *croissance exponentielle ξ-uniforme uniforme* si chaque sous-groupe de type fini est soit virtuellement nilpotent, soit a croissance exponentielle ξ-uniforme.

Example 0.3.1. — Voici des familles de groupes à croissance exponentielle uniforme uniforme agissant de manière acylindrique sur des espaces hyperboliques :

(i) Groupes hyperboliques.

(ii) Produits libres de *familles dénombrables* de groupes à croissance exponentielle ξ-uniforme uniforme.

(iii) Quelques groupes cubiques CAT(0).

(iv) Groupes modulaires des surfaces.

En général, le paramètre de croissance uniforme dépend du groupe.

Vers une théorie géométrique des petite simplification. Soit G un groupe agissant par isométries sur un espace δ-hyperbolique X. Une *famille de mouvement* – ou *ensemble de relations* – est un ensemble de la forme

$$\mathscr{Q} = \left\{ \left(\langle grg^{-1} \rangle, gY_r \right) \;\middle|\; r \in \mathscr{R},\ g \in G \right\},$$

où $\mathscr{R} \subset G$ est un ensemble d'isométries loxodromiques r – les *relateurs* – stabilisant leur axe quasi-convexe $Y_r \subset X$. Une *pièce* est une intersection de n'importe quelle paire de tels axes. Le rôle des paramètres $\lambda \in (0, 1)$ et $\varepsilon > 0$ dans la condition de petite simplification géométrique $C''(\lambda, \varepsilon)$ sur $\mathscr{Q}$ est le suivant:

- La fraction de la *longueur de la pièce la plus long* avec la *longueur de translation la plus courte des relations* $r \in \mathscr{R}$ est au plus λ.

- ▶ La *longueur de translation la plus courte des relations* $r \in \mathscr{R}$ est au moins $\varepsilon\delta$.

Soit K la clôture normale dans G des *sous-groupes de relation* H dans $\mathscr{Q}$. La condition de petite simplification géométrique $C''(\lambda, \varepsilon)$ permet d'obtenir des informations substantielles sur le quotient à $C''(\lambda, \varepsilon)$-petite simplification géométrique $\bar{G} = G/K$: par exemple K est un produit libre de sous-groupes de relation, $\bar{G}$ ressemble localement à G et toute action acylindrique de G sur X induit une autre action acylindrique de $\bar{G}$ sur un quotient δ_0-espace hyperbolique $\bar{X}$ dont la constante d'hyperbolicité δ_0 est universelle.

Le résultat principal du Chapitre 2 est le suivant:

Theorem 0.3.2 (Theorem 2.5.5 & Theorem 2.5.6)**. —** *Il existe $\lambda \in (0, 1)$ tel que pour chaque $N > 0$ et $\varepsilon > 10^{10}N$, ce qui suit est vrai. Soient $\delta > 0$, $\kappa \geqslant \delta$ et soit G un groupe agissant (κ, N)-acylindriquement sur un espace δ-hyperbolique X.*

(i) Si G est à croissance exponentielle ξ-uniforme uniforme, alors chaque quotient à $C''(\lambda, \varepsilon)$-petite simplification géométrique de G est à croissance exponentielle ξ'-uniforme uniforme. La constante ξ' ne dépend que de ξ et N.

(ii) S'il existe un quotient à $C''(\lambda, \varepsilon)$-petite simplification géométrique de G qui est à croissance exponentielle ξ-uniforme uniforme, alors G est à croissance exponentielle ξ'-uniforme uniforme. La constante ξ' ne dépend que de ξ.

Au-delà de la propriété du élément loxodromique court. La stratégie standard pour étudier la croissance exponentielle uniforme dans les groupes hyperboliques exploite le fait que leurs sous-ensembles générateurs ont la *propriété de l'élément loxodromique court* : chaque n-ième puissance U^n d'un sous-ensemble générateur fini contient une isométrie loxodromique, pour un certain nombre n qui ne dépend pas de l'ensemble U. En général, on ne sait pas si chaque groupe de type fini agissant de manière acylindrique sur un espace hyperbolique a une croissance exponentielle uniforme. L'action acylindrique sur un espace hyperbolique donne une croissance exponentielle uniforme pour des sous-ensembles générateurs finis avec une longue isométrie loxodromique. La propriété de l'élément loxodromique court permet de prendre des grandes puissances uniformes pour pouvoir exploiter cette autre situation. Cependant, il y a un quotient à petite simplification *combinatoire/gradué* avec une action acylindrique sur un espace hyperbolique mais sans la propriété de l'élément loxodromique court, [63]. Notre résultat principal ne fait pas usage de la propriété de l'élément loxodromique court. La morale de notre travail est que nous pouvons traiter ce genre de monstre tant que ce sont des quotients à petite simplification de groupes de croissance exponentielle uniforme uniforme agissant de manière acylindrique sur un espace hyperbolique. Cependant, le monstre mentionné est un quotient du produit libre de tous les groupes hyperboliques. On ne sait pas s'il existe une borne inférieure universelle pour le taux de croissance uniforme dans la classe de tous les groupes hyperboliques, indépendante de la constante d'hyperbolicité, c'est une autre question ouverte. Ça équivaut à la croissance exponentielle uniforme uniforme du produit libre de tous les groupes hyperboliques.

Introduction

Kill the boy, Jon Snow. Winter is almost upon us.
Kill the boy and let the man be born.

from *A Dance with Dragons* of George R. R. Martin,
spoken by Maester Aemon

Contents

0.4 Abstract

The focus of this thesis is on questions that compare numbers easy to define but not easy to compute. The action of a group G on a metric space X is called *proper* if for every $r > 0$, and for every $x \in X$, the number of elements $u \in G$ moving x at distance at most r is finite. Let G be a group acting properly by isometries on a metric space X. The *relative exponential growth rate* of the action of a subset $U \subset G$ on X is the number

$$\omega(U, X) = \limsup_{r \to \infty} \frac{1}{r} \log |\{ u \in U \ : \ |ux - x| \leqslant r \}|,$$

whose value is independent of the point $x \in X$. If G is the fundamental group of a closed hyperbolic manifold M acting on the universal cover X, then $\omega(G, X)$ has numerous interpretations. It coincides with the *volume entropy* of the manifold M, [71, 62]; the *critical exponent* of the Poincaré series of G, [67, 76]; the *topological entropy* of the geodesic flow on the unit tangent bundle of M, [60]; the *Hausdorff dimension* of the radial limit set of G, [28], etc. In this context, the number $\omega(G, X)$ is the cornerstone bringing together groups, geometry and dynamics. The discreteness of the orbit of G and the negative curvature of M play a major role in this phenomenon.

The aim of this thesis is to quantify growth in groups from their actions by isometries on metric spaces. The approach is to observe from far away. The exploit is on the finiteness and non-positive curvature conditions of actions and spaces. Let δ, κ, $N > 0$. The action of a group G on a δ-hyperbolic space X is called (κ, N)-*acylindrical*, [72, 18, 66, 38], if for every pair of points $x, y \in X$ at distance at least κ, the number of elements $u \in G$ moving each of the points x, y at distance at most 100δ is bounded above by N. In the last decade, the focus has been put on groups that admit an acylindrical action on a Gromov hyperbolic space, [66]. This is a vast family of groups that includes relatively hyperbolic groups, infinitely presented classical small cancellation groups, mapping class groups of surfaces, outer automorphism groups of free groups, right angled Artin groups, etc. It turns out that most of the time groups acting acylindrically on a hyperbolic space admit remarkable proper actions on other spaces that are not necessarily hyperbolic, but contain isometries that behave as the loxodromic isometries of a hyperbolic space: *constricting elements*, [74], under the terminology of [8]. In fact, the converse is always true.

The moral of the thesis draws the following idea of M. Gromov: under a non-positively curved global viewpoint, it is still possible to produce strong results for a typical group, which can sometimes approximate our understanding to monster groups. We are going to study two different problems using low tech arguments involving the triangle inequality. The first one will be about the growth of quasi-convex subgroups in groups acting properly with a constricting element. In addition, we have added an appendix describing some elementary consequences of the constriction property. The second will be about the uniform growth in small cancellation quotients over groups acting acylindrically on a hyperbolic space. Chapters 1 and 2 correspond respectively to the following articles:

- X. Legaspi. Constricting elements and the growth of quasi-convex subgroups, 2022. URL: https://orcid.org/0000-0002-1497-6448.
- X. Legaspi and M. Steenbock. Uniform uniform exponential growth in small cancellation groups, 2023. URL: https://orcid.org/0000-0002-1497-6448.

0.5 Growth of quasi-convex subgroups

Let G be a group acting properly by isometries on a metric space X. Let $x \in X$. Let H be a subgroup of G. Let H_L and H_R be respectively left and right transversals of H

such that for every $u \in H_L$ and $v \in H_R$,

$$|ux - x| = \inf_{h \in H} |uhx - x|, \quad \text{and } |vx - x| = \inf_{h \in H} |hvx - x|.$$

In Chapter 1 we study the numbers

$$\omega(H) := \omega(H, X), \quad \omega(G/H) := \omega(H_L, X), \quad \text{and } \omega(H\backslash G) := \omega(H_R, X).$$

Consider the following general problem. When do G and H determine a solution to to the system of equations below?

$$\begin{cases} \omega(H) < \omega(G), \\ \omega(G/H) = \omega(G), \\ \omega(H\backslash G) = \omega(G). \end{cases}$$

We see from the definitions that

$$\omega(H/G) = \omega(H\backslash G), \quad \text{and } 0 \leqslant \max\{\omega(H), \omega(G/H)\} \leqslant \omega(G).$$

In the extreme case in which H has finite index in G, one can easily prove that

$$\begin{cases} \omega(H) = \omega(G), \\ \omega(G/H) = 0. \end{cases}$$

In general, it is a hard problem to obtain precise estimations of relative exponential growth rates of infinite index subgroups. However, it is known, [37, 4, 47], that if G is a non-virtually cyclic group acting geometrically on a hyperbolic space X and H is an infinite index quasi-convex subgroup of G, then

$$\begin{cases} \omega(H) < \omega(G), \\ \omega(G/H) = \omega(G). \end{cases}$$

The arguments of [37, 4] are based on automatic structures and regular languages, with influence of the works of J. Cannon [22, 23]. This fact also influenced other authors that partially extended the hyperbolic case result, [27]. In Chapter 1 we go beyond the hyperbolic case and we obtain two main results (Theorem 0.5.8 and Theorem 0.5.13) with elementary proofs that do not require the theory of regular languages and automata. We will be interested in groups acting properly on metric spaces conditioned by a very general notion of "non-positive curvature" introduced by A. Sisto in [74] — *containing a constricting element with respect to a path system* — while the infinite index subgroups object of our study will satisfy a very general notion of "convex cocompactness" — *quasi-convexity with respect to a path system.*

The remaining of this section is structured as follows. First of all, we will mention two applications. Later we will give an informal explanation of our general setting as the result of a natural generalisation of these applications. We expect that this will be enough to understand our main theorems stated right after that. We will give another application at the end.

Groups acting properly with a strongly contracting element. Members of this class contain elements that "behave like" a loxodromic isometry in a hyperbolic space – in a strong sense. Let $\delta \geqslant 0$. A *subset* A of X is δ-*strongly contracting* if the diameter of the nearest-point projection on A of any metric ball of X not intersecting A is less than δ. An *element* g of G is δ-*strongly contracting* if it has infinite order and there exists an orbit of the cyclic subgroup generated by g that is δ-strongly contracting. In his seminal paper M. Gromov introduced the concept of δ-hyperbolic space, [49]. He observed that most of the large scale features of negative curvature can be described in terms of thin triangles. Nowadays, there are plenty of reformulations of the δ-hyperbolicity. In particular, H. Masur and Y. Minsky gave one by describing geodesics in terms of strong contraction:

Example 0.5.1. — A geodesic metric space X is hyperbolic if and only if there exists $\delta \geqslant 0$ such that any geodesic segment of X is δ-strongly contracting, [61, Theorem 2.3].

The following are some subclasses of groups acting properly with a strongly contracting element:

(i) $\mathbf{H}$ = "G is a group acting properly with a loxodromic element on a hyperbolic space X." In $\mathbf{H}$, an element is loxodromic if and only if it is strongly contracting. See [26].

(ii) $\mathbf{RH}$ = "G is a relatively hyperbolic group acting with a hyperbolic element on a locally finite Cayley graph X of G." In $\mathbf{RH}$, hyperbolic elements are strongly contracting. See [65, Corollary 1.7] and [73, Theorem 2.14].

(iii) $\mathbf{CAT_0}$ = "G is a group acting properly with a rank-one element on a proper $CAT(0)$ space X." In $\mathbf{CAT_0}$, rank-one elements are strongly contracting. See [16, Theorem 5.4] and [24].

(iv) $\mathbf{Mod_T}$ = "G is the mapping class group of an orientable surface of genus g and p marked points of complexity $3g + p - 4 > 0$ acting on its Teichmüller space endowed with the Teichmüller metric." In $\mathbf{Mod_T}$, pseudo-Anosov elements are strongly contracting. See [64] and [61, Proposition 4.6].

(v) **GSC** = "G is an infinite graphical small cancellation group associated to a $Gr'(1/6)$-labeled graph with finite components labeled by a finite set S, acting on the Cayley graph X of G with respect to S." In **GSC**, loxodromic WPD elements for the action of G on the hyperbolic coned-off Cayley graph constructed by D. Gruber and A. Sisto in [51] are strongly contracting. See [7, Theorem 5.1].

(vi) **Gar** = "G is the quotient of a Δ-pure Garside group of finite type by its center, acting with a Morse element on the Cayley graph X of G with respect to the Garside generating set." In **Gar**, Morse elements are strongly contracting. See [21, Theorem 5.5].

(vii) **Inj** = "G is a group acting properly with a Morse element on an injective metric space X." In **Inj**, an element is Morse if and only if it is strongly contracting. See [75].

An appropriate notion of convex cocompactness in this setting is just the usual quasi-convexity. Let $\eta \geqslant 0$. A *subset* Y of X is η-quasi-convex if any geodesic of X with endpoints in Y is contained in the η-neighbourhood of Y. A *subgroup* H of G is *η-quasi-convex* if there exists an orbit of H that is η-quasi-convex.

Our theorem below generalises [78, Theorem 4.8] and [37, Theorems 1.1 and 1.3]:

Theorem 0.5.2. — *If G is a non-virtually cyclic group acting properly with a strongly contracting element on a geodesic metric space X, and H is an infinite index quasi-convex subgroup of G, then*

$$\begin{cases} \omega(H) < \omega(G), \\ \omega(G/H) = \omega(G). \end{cases}$$

Hierarchically hyperbolic groups. Let $\mathrm{Mod}(\Sigma_{g,p})$ be the mapping class group of an orientable surface $\Sigma_{g,p}$ of genus g and p marked points of complexity $3g+p-4>0$. We would like to apply Theorem 0.5.2 to $\mathrm{Mod}(\Sigma_{g,p})$ with respect to the word metric. However, we do not know whether $\mathrm{Mod}(\Sigma_{g,p})$ acts with a strongly contracting element on any of its locally finite Cayley graphs or not. Maybe the candidates that come to mind are the pseudo-Anosov elements, and evidence suggests that not all of them are strongly contracting: K. Rafi and Y. Verberne constructed a generating set U of $\mathrm{Mod}(\Sigma_{0,5})$ and a pseudo-Anosov element which is not strongly contracting for the action of $\mathrm{Mod}(\Sigma_{0,5})$ on the Cayley graph of $\mathrm{Mod}(\Sigma_{0,5})$ with respect to U, [68, Theorem 1.3]. We were able to avoid this setback by looking into the class of hierarchically hyperbolic groups, introduced by J. Behrstock, M. Hagen and A.Sisto in [12, 13] as a generalisation of the Masur and Minsky hierarchy machinery of mapping class groups. Below we provide some examples of hierarchically hyperbolic groups. The reader should note that the metric space where they act with a hierarchically hyperbolic structure is any of their locally finite Cayley graphs:

(i) Mapping class groups of finite type surfaces, [13].

(ii) Right-angled Artin groups, [12].

(iii) Right-angled Coxeter groups, [12].

(iv) Fundamental groups of 3-manifolds without NIL or SOL components, [13].

Now consider the following notion of convex cocompactness. A *subset* Y of X is *Morse* if for every $\kappa \geqslant 1$, $\lambda \geqslant 0$, there exists $\sigma \geqslant 0$ such that any (κ, l)-quasi-geodesic of X with endpoints in Y is contained in the σ-neighbourhood of Y. A *subgroup* H of G is *Morse* if there exists an orbit of H that is Morse. An *element* g of G is *Morse* if it has infinite order and the cyclic subgroup generated by g is Morse.

We have obtained the next result, partially generalising [27, Theorem A]:

Theorem 0.5.3. — *If G is a non-virtually cyclic hierarchically hyperbolic group acting on a locally finite Cayley graph X of G with a Morse element, and H is an infinite index Morse subgroup of G, then*

$$\begin{cases} \omega(H) < \omega(G), \\ \omega(G/H) = \omega(G). \end{cases}$$

We know that pseudo-Anosov elements of mapping class groups are Morse with respect to any word metric, [11], and that the infinite index Morse subgroups of the mapping class group are precisely the convex cocompact subgroups in the sense of mapping class groups, [55, Theorem A], which allows us to obtain a more concrete statement:

Corollary 0.5.4. — *If G is the mapping class group of a surface of genus g and p marked points such that $3g + p - 4 > 0$ acting on a locally finite Cayley graph X of G, and H is a* *convex cocompact subgroup of G, then*

$$\begin{cases} \omega(H) < \omega(G), \\ \omega(G/H) = \omega(G). \end{cases}$$

Remark 0.5.5. — Under the hypothesis of the previous corollary, we remark that the inequality $\omega(H) < \omega(G)$ was also obtained independently in [27, Corollary C].

Main results. Now that we gave the big picture, we will give a technical definition that encapsulates the classes discussed so far. In order to do so, we make two observations. On the one hand, the strong contraction property can be reformulated in the following way. A subset A of X is *strongly contracting* if and only if any geodesic segment of X joining any pair of points $x, y \in X$ whose projections p and q via a nearest-point projection are far away passes next to p and q, [8, Proposition 2.9]. On the other hand, mapping class groups – or more generally, hierarchically hyperbolic groups – come with hierarchy paths, a family of special quasi-geodesics encoding substantial information about the geometry of the space and easier to work with than the set of all (quasi-)geodesics. For these reasons, in order to define very general notions of non-positive curvature and convex cocompactness, we will be considering path systems, introduced by A. Sisto in [74]:

Definition 0.5.6 (Path system group)**.** — Let $\mu \geqslant 1$, $\nu \geqslant 0$. A (μ,ν)-*path system group* $(G, X, \mathscr{P})$ is a group G acting properly on a geodesic metric space X together with a G-invariant collection $\mathscr{P}$ of paths of X satisfying:

(PS1) $\mathscr{P}$ is closed under taking subpaths.

(PS2) For every $x, y \in X$, there exists $\gamma \in \mathscr{P}$ joining x to y.

(PS3) Every element of $\mathscr{P}$ is a (μ,ν)-quasi-geodesic.

We refer to $\mathscr{P}$ as (μ,ν)-*path system*.

We fix $\mu \geqslant 1$, $\nu \geqslant 0$ and a (μ,ν)-path system group $(G, X, \mathscr{P})$ for the following definitions. Let $\delta \geqslant 0$. We say that a subset A of X is δ-*constricting* if there exist a *coarse nearest-point projection* of X on A with the property that any $\gamma \in \mathscr{P}$ joining any two pair of points $x, y \in X$ whose projections p and q are δ-far away passes through the δ-neighbourhoods of p and q (Definition 1.1.8). An *element* g of G is δ-*constricting* if it has infinite order and there exists a δ-constricting orbit of the cyclic subgroup generated by g. Let $\eta \geqslant 0$. A subgroup Y of X is η-*quasi-convex* if any $\gamma \in \mathscr{P}$ with endpoints in

Y is contained in the η-neighbourhood of Y (Definition 1.1.7). A subgroup H of G is η-quasi-convex if there exist an η-quasi-convex orbit of H.

Example 0.5.7. — (i) Assume that the metric space X is geodesic. An infinite order element of G is strongly contracting if and only if it is constricting with respect to the set of all the geodesic segments of X, [8, Proposition 2.9].

(ii) Assume that the group G is hierarchically hyperbolic. An infinite order element g of G is Morse if and only if for every $\kappa \geqslant 1$, there exists $\delta \geqslant 0$ such that g is δ-constricting with respect to the set of all the κ-hierarchy paths. See [69, Theorem E] and [14, Lemma 1.27].

Finally, we state the main results of Chapter 1. Theorem 0.5.2 and Theorem 0.5.3 are special cases. Our first result generalises work of W. Yang, [78, Theorem 4.8], and F. Dahmani - D. Futer - D. Wise, [37, Theorems 1.1 and 1.3]. The *Poincaré series* $\mathscr{P}_U(s)$ based at $x \in X$ of a subset U of G is defined as

$$\forall\, s \geqslant 0, \quad \mathscr{P}_U(s) = \sum_{u \in U} e^{-s|ux - x|}$$

and modifies its behaviour at the relative exponential growth rate $\omega(U, X)$: the series diverges if $s < \omega(U, X)$ and converges if $s > \omega(U, X)$. At $s = \omega(U, X)$ the series can converge or diverge depending on the nature of U. This behaviour is independent of the point $x \in X$. We say that the action of U on X is *divergent* if $\mathscr{P}_U(s)$ diverges at $s = \omega(U, X)$.

Theorem 0.5.8. — *Let $(G, X, \mathscr{P})$ be a path system group. Assume that G contains a constricting element. Let H be an infinite index subgroup of G. Assume that the following conditions are true:*

(i) $\omega(H) < \infty$.

(ii) The action of H on X is divergent.

(iii) H is quasi-convex.

Then $\omega(H) < \omega(G)$.

Remark 0.5.9. — Under the hypothesis of Theorem 0.5.8, one may ask if there is a growth gap, i.e, if

$$\sup_H \omega(H) < \omega(G),$$

where the supremum is taken among the infinite index subgroups H of G satisfying (i), (ii) and (iii). In our context, the answer is yes: there is a growth gap when G is a hyperbolic group with *Kazhdan's Property* (T), [34, Theorem 1.2]. However, one can show that there is no growth gap among free groups, [37, Theorem 9.4], or fundamental groups of compact special cube complexes, [58, Theorem 1.5]. The answer to our context could be different if one studied semigroups instead of subgroups, [78, Theorem A].

In [49, 5.3.C], M. Gromov stated that in a torsion-free hyperbolic group G, any infinite index quasi-convex subgroup H is a free factor of a larger quasi-convex subgroup. Gromov's ideas were later developed by G. N. Arzhantseva in [6, Theorem 1]. More recently, J. Russell, D. Spriano and H. C. Tran generalised her result to the context of groups with the "Morse local-to-global property", [70, Corollary 3.5]. Further, the problem seems connected to the "P_{Naive} property" studied by C. Abbott and F. Dahmani in the context of groups acting acylindrically on a hyperbolic space, [1]. In our context, we have obtained the following, in which there is no torsion-free assumption. We will see that Theorem 0.5.8 is, in part, a consequence of this result:

Theorem 0.5.10. — *Let $(G, X, \mathscr{P})$ be a path system group. Assume that G contains a constricting element g_0. Let H be an infinite index quasi-convex subgroup of G. There exist an element $g \in G$ conjugate to a large power of g_0 and a finite extension E of $\langle g\rangle$ such that the intersection $H \cap E$ is finite and the natural morphism $H *_{H\cap E} \langle g, H \cap E\rangle \to G$ is injective.*

According to Proposition 1.1.5 (6), the subgroup generated by a constricting element is always Morse, and in particular quasi-convex. Hence we obtain the following alternative:

Corollary 0.5.11. — *Let $(G, X, \mathscr{P})$ be a path system group. Assume that G contains a constricting element. Then, either G is virtually cyclic or contains a free subgroup of rank two.*

Remark 0.5.12. — To the best of our knowledge, the previous corollary has not been recorded for the class of groups acting properly with a strongly contracting element. The Tits alternative is known for hierarchically hyperbolic groups [43, Theorem 9.15], which is a much stronger result.

In our second result we generalise work of Y. Antolín, [4, Theorem 3], and R. Gitik - E. Rips, [47, Theorem 2]:

Theorem 0.5.13. — *Let $(G, X, \mathscr{P})$ be a path system group. Assume that G contains a constricting element. Let H be an infinite index quasi-convex subgroup of G. Then*

$$\omega(G/H) = \omega(G).$$

Note that the study of [47, Theorem 2] concerns double cosets in the hyperbolic group case. We remark that in [40, VII D 39], P. de la Harpe says about the growth of double cosets: "this theme has not received yet too much attention, but probably should". In our context, for sake of simplicity, we decided to study single cosets instead, but one could possibly extend our result. Further, we remark that our result is connected to the study of I. Kapovich on the hyperbolicity and amenability of the Schreier graphs of infinite index quasi-convex subgroups of hyperbolic groups, [53, 54].

Now we are going to record a joint corollary to Theorem 0.5.8 and Theorem 0.5.13. In general, it is not easy to decide whether the action of a groups is divergent or not. However, the following is a well-known consequence of *Fekete's Subadditivity Lemma*:

Lemma 0.5.14 ([39, Proposition 4.1 (1)]). **—** *Let G be a group acting properly on a geodesic metric space X. Let $x \in X$. Let $H \leqslant G$ be a quasi-convex subgroup (in the classical sense). Then*

$$\omega(H) = \inf_{n \geqslant 1} \frac{1}{n} \log |\{\, h \in H \,:\, |hx - x| \leqslant n \,\}| = \lim_{n \to \infty} \frac{1}{n} \log |\{\, h \in H \,:\, |hx - x| \leqslant n \,\}|.$$

In particular $\omega(H) < \infty$. If in addition H is infinite, then the action of H on X is divergent.

In combination with Corollary 0.5.11, we obtain:

Corollary 0.5.15. — *Let $(G, X, \mathscr{P})$ be a path system group. Assume that G is non-virtually cyclic and contains a constricting element.*

(i) If $\mathscr{P}$ is the set of all the geodesic segments of X, then for every infinite index quasi-convex subgroup H of G, we have

$$\begin{cases} \omega(H) < \omega(G), \\ \omega(G/H) = \omega(G). \end{cases}$$

(ii) For every infinite index Morse subgroup H of G, we have

$$\begin{cases} \omega(H) < \omega(G), \\ \omega(G/H) = \omega(G). \end{cases}$$

Remark 0.5.16. — One can prove that the class of groups acting properly with a constricting element with respect to a path system is invariant under equivariant quasi-isometries. However, strongly contracting elements are not preserved under equivariant quasi-isometries, [7, Theorem 4.19]. In particular, Corollary 0.5.15 applies for instance to the action on a locally finite Cayley graph of any group acting geometrically on a $CAT(0)$ space with a rank-one element.

Remark 0.5.17. — The proofs of Theorem 0.5.2 and Theorem 0.5.3 now follow from our main results (Theorem 0.5.8 and Theorem 0.5.13) in view of Example 0.5.7.

Hierarchical quasi-convexity. In hierarchically hyperbolic groups there is a notion of convex cocompactness more natural than Morseness. Let G be a hierarchically hyperbolic group. A subgroup H of G is hierarchically quasi-convex if and only if for every $\kappa \geqslant 1$, there exists $\eta \geqslant 0$ such that H is η-quasi-convex with respect to the set of all the κ-hierarchy paths of G, [69, Proposition 5.7]. Finally, we deduce two more applications from Theorem 0.5.8 and Theorem 0.5.13:

Theorem 0.5.18. — *If G is a hierarchically hyperbolic group acting on a locally finite Cayley graph X of G with a Morse element, and H is an infinite index subgroup of G satisfying:*

(i) The action of H on X is divergent.

(ii) H is hierarchically quasi-convex.

Then $\omega(H) < \omega(G)$.

Theorem 0.5.19. — *If G is a hierarchically hyperbolic group acting on a locally finite Cayley graph X of G with a Morse element, and H is an infinite index hierarchically quasi-convex subgroup of G, then $\omega(G/H) = \omega(G)$.*

0.6 Uniform uniform exponential growth

Let G be a group with finite symmetric generating set U. Denote by X_U the corresponding Cayley graph. In Chapter 2 we study the number

$$\omega(U) := \omega(G, X_U).$$

The n-th product set U^n is the collection of elements $u_1 \cdot \ldots \cdot u_n \in G$ such that $u_1, \cdots, u_n \in U$. The role of $\omega(U)$ is to give us information about the exponential behaviour of $|U^n|$ as n increases. The generating sets of virtually nilpotent groups have vanishing exponential growth rate, since a celebrated theorem of M. Gromov shows that those are exactly the groups of polynomial growth, [48]. Let $\xi > 0$. The group G has *ξ-uniform exponential growth* if for every finite symmetric generating set U of G, we have $\omega(U) > \xi$. A group has *ξ-uniform uniform exponential growth* if every finitely generated subgroup is either virtually nilpotent or has ξ-uniform exponential growth.

Uniform uniform exponential growth is particularly well-studied in groups of non-positive curvature. Indeed, groups of uniform uniform exponential growth include hyperbolic groups, [56, 9, 19], free products of countable families of groups with ξ-uniform uniform exponential growth (folklore), mapping class groups, [3, 59, 2], or cocompactly special cubulated CAT(0) groups, [44, 2]. It is unknown whether the outer automorphism group of the free group of rank $\geqslant 2$ has uniform uniform exponential growth, [15]. All of the groups in this list admit non-elementary acylindrical actions on Gromov hyperbolic spaces, [72, 17, 38].

Geometric small cancellation quotients. The main goal of Chapter 2 is to prove that the class of groups of uniform uniform exponential growth acting acylindrically on a hyperbolic space is closed under taking geometric $C''(\lambda, \varepsilon)$-small cancellation quotients in the sense of [38, Definition 6.22]. This result is Theorem 0.6.2 below. Before stating the theorem, we are going to give some definitions. Let $\delta > 0$. Let G be a group acting by isometries on a δ-hyperbolic space X.

Acylindricity. Let κ, $N > 0$. The action of G on X is (κ, N)-*acylindrical* if for every pair of points $x, y \in X$ at distance at least κ, the number of elements $u \in G$ moving each of the points x, y at distance at most 100δ is bounded above by N. In practice, the number N has two meanings for us:

(1) The largest size of the finite subgroups of virtually cyclic subgroups in G containing

a loxodromic isometry.

(2) The fraction $\frac{\Delta(g)}{\|g\|}$ of the longest intersection $\Delta(g)$ between the axis of any pair of conjugates of an arbitrary loxodromic isometry g of G, with the *translation length* $\|g\|$ of g, whenever this translation is larger than 100δ.

Geometric small cancellation theory. A *loxodromic moving family* – or *set of relations* – is a set of the form

$$\mathscr{Q} = \left\{ \left(\langle grg^{-1} \rangle, gY_r \right) \;\middle|\; r \in \mathscr{R},\ g \in G \right\},$$

where $\mathscr{R} \subset G$ is a set of loxodromic isometries r – the *relators* – stabilizing their quasi-convex axis $Y_r \subset X$. A *piece* is an intersection of any pair of such axis. The role of the parameters $\lambda \in (0,1)$ and $\varepsilon > 0$ in the geometric $C''(\lambda,\varepsilon)$-small cancellation condition on $\mathscr{Q}$ is the following:

- The fraction of the *length of the longest piece* with the *shortest translation length of the relators* $r \in \mathscr{R}$ is at most λ.
- The *shortest translation length of the relators* $r \in \mathscr{R}$ is at least $\varepsilon\delta$.

Let K be the normal closure in G of the *relator subgroups* H in $\mathscr{Q}$. The geometric $C''(\lambda,\varepsilon)$-small cancellation condition permits to obtain substantial information of the geometric $C''(\lambda,\varepsilon)$-small cancellation quotient $\bar{G} = G/K$: for instance K is a free product of relator subgroups, $\bar{G}$ locally looks like G and any acylindrical action of G on X induces another acylindrical action of $\bar{G}$ on a quotient $\bar{\delta}$-hyperbolic space $\bar{X}$ whose hyperbolicity constant $\bar{\delta}$ is universal.

Main theorem. The following corollary captures the essence of the main theorem.

Corollary 0.6.1. — *There exists a universal constant $\lambda > 0$ such that for every group G acting acylindrically on a hyperbolic space X, there exist $\varepsilon > 0$ depending only on the acylindricity and hyperbolicity constants such that the following statements are equivalent.*

(i) G has uniform uniform exponential growth.

(ii) Every geometric $C''(\lambda,\varepsilon)$-small cancellation quotient of G has uniform uniform exponential growth.

(iii) There exists a geometric $C''(\lambda,\varepsilon)$-small cancellation quotient of G that has uniform uniform exponential growth.

The main theorem of Chapter 2 is the following.

Theorem 0.6.2 (Theorem 2.5.5 & Theorem 2.5.6)**. —** *There exists* $\lambda > 0$ *such that for every* $N > 0$ *the following holds. Let* $\delta > 0$*,* $\kappa \geqslant \delta$*, and* $\varepsilon \geqslant 10^{10} \max\{N, \kappa/\delta\}$*. Let* G *be a group acting* (κ, N)*-acylindrically on a* δ*-hyperbolic space* X*.*

(i) If G *has* ξ*-uniform uniform exponential growth, then every geometric* $C''(\lambda, \varepsilon)$*-small cancellation quotient of* G *has* ξ'*-uniform uniform exponential growth. The constant* ξ' *depends only on* ξ *and* N*.*

(ii) If there exist a geometric $C''(\lambda, \varepsilon)$*-small cancellation quotient of* G *that has* ξ*-uniform uniform exponential growth, then* G *has* ξ'*-uniform uniform exponential growth. The constant* ξ' *depends only on* ξ*.*

Remark 0.6.3. — The dependence of ε on κ, N and δ is not a strong condition. In fact, the intersection of the axis of two loxodromic elements in a group acting acylindrically on a hyperbolic space is controled in terms of κ, N, δ and the translation length of the loxodromic elements. Thus to prove that a set of relators satisfies the geometric $C''(\lambda, \varepsilon)$-condition, one usually considers relators of sufficient length compared to κ, N and δ anyway.

Beyond short loxodromics. The standard strategy to study uniform exponential growth in hyperbolic groups exploits the fact that their finite symmetric generating sets have the *short loxodromic property*: every n-th power U^n of a finite symmetric generating set contains a loxodromic isometry, for some number n that does not depend on the set U. In general, it is unknown whether every finitely generated group acting acylindrically on a hyperbolic space has uniform exponential growth. The acylindrical action on a hyperbolic space yields uniform exponential growth for finite symmetric generating sets with a long loxodromic isometry. The short loxodromic property permits to take uniform large powers so that we can exploit this other situation. However, there is a finitely generated (*combinatorial/graded*) small cancellation quotient with an acylindrical action on a hyperbolic space but without the short loxodromic property, [63]. Our main result does not make use of the short loxodromic property. The moral of our work is that we can deal with this kind of monster *as long as* these are small cancellation quotients of groups of uniform uniform exponential growth acting acylindrically on a hyperbolic space. However, the aforementioned monster is a quotient of the free product of all hyperbolic groups. It is unkown whether this free product has uniform uniform exponential growth, owing to it is unkown whether there is a universal lower bound for the uniform growth rate of all hyperbolic groups, independent of the hyperbolicity constant, [19, Section 14, Question 2]. The following example shows that the short loxodromic property plays no role in the proof of Theorem 0.6.2.

Example 0.6.4. — There are infinite families of geometric small cancellation quotients that are hyperbolic groups containing arbitrarily large torsion balls. These groups act acylindrically with uniform acylindricity parameters and have ξ-uniform uniform exponential growth, for some uniform growth exponent $\xi > 0$, see [36]. The uniform uniform exponential growth rate of the small cancellation quotient in Theorem 0.6.2 (i) does not depend on the cardinality of large torsion balls, nor does it depend on the hyperbolicity constant.

Classical small cancellation groups We now discuss groups given by a presentation that satisfies the classical $C''(\lambda)$-small cancellation condition. We refer to a group admiting such a presentation as classical $C''(\lambda)$-small cancellation group. These are exacly the geometric small cancellation quotients over free groups. In this situation, the geometric small cancellation condition involving the parameter ε becomes trivial. A classical $C''(\lambda)$-small cancellation group is always finitely presented, hence, hyperbolic. Thus it has uniform uniform exponential growth by [49, 56]. However, in that approach the uniform uniform exponential growth rate depends on λ. The following is a consequence of Theorem 0.6.2 for the free group case.

Corollary 0.6.5. — *There exist $\lambda > 0$ and $\xi > 0$ such that every classical $C''(\lambda)$-small cancellation group has ξ-uniform uniform exponential growth.*

Note that there is a generic class of classical $C''(1/6)$-small cancellation groups such that every 2-generated subgroup is free, [10]. This immediately implies Corollary 0.6.5 for this generic class of classical $C''(1/6)$-small cancellation groups, [40].

Remark 0.6.6. — The classical $C''(\lambda)$-small cancellation condition in Corollary 0.6.5 is reminiscent of our proof that uses geometric small cancellation theory. To this date, geometric small cancellation theory has not been developed under a geometric $C'(\lambda, \varepsilon)$-small cancellation condition. We expect, however, that this is possible, and thus that our results hold for classical $C'(\lambda)$-small cancellation groups - finitely and infinitely presented.

Strategy of proof. To prove Theorem 0.6.2 (i), we need to discuss the growth of finite symmetric subsets of sufficiently large energy in groups acting acylindrically on a hyperbolic space X. If G acts by isometries on X, the ℓ^∞-energy $\mathrm{L}(U)$ of a finite subset $U \subset G$ is defined by

$$\mathrm{L}(U) = \inf_{x\in X} \max_{u\in U} |ux - x|.$$

If $U = \{g\}$, the ℓ^∞-energy coincides with the translation length of g. The following example explains why the energy is important in the study of uniform exponential growth.

Example 0.6.7. — When G is the fundamental group of a compact hyperbolic manifold, there exists a constant $\mu > 0$ – the *Margulis constant* – such that if $U \subset G$ is a finite set with $\mathrm{L}(U) < \mu$, then the subgroup of G generated by U is virtually nilpotent. If T denotes the injectivity radius of the action of G on the universal cover and is smaller than the Margulis constant μ, then the acylindricity constant κ is about $1/\mathrm{T}$, [42].

Definition 0.6.8 (Definition 2.2.1)**. —** Let $\alpha > 0$. We say that a finite subset $U \subset G$ is *α-reduced at* $p \in X$ if $U \cap U^{-1} = \varnothing$ and for every pair of distinct $u_1, u_2 \in U \sqcup U^{-1}$, the Gromov product satisfies

$$(u_1p, u_2p)_p < \frac{1}{2}\min\{|u_1p - p|, |u_2p - p|\} - \alpha - 2\delta.$$

Remark 0.6.9. — Roughly speaking, if a set $U \subset G$ is reduced then the orbit map from the free group generated by U to X is a quasi-isometric embedding.

The following is a well-known theorem of [56, 9], see also [45].

Theorem 0.6.10 (Theorem 2.3.8)**. —** *For every κ, $N > 0$, there exist an integer $c > 1$ with the following property. Let δ, $\alpha > 0$. Let G be a group acting (κ, N)-acylindrically on a δ-hyperbolic space X. Let $U \subset G$ be a finite symmetric subset containing the identity. Then one of the following conditions holds:*

(i) $\mathrm{L}(U) \leqslant 10^4 \max\{\kappa, \delta, \alpha\}$.

(ii) The subgroup $\langle U\rangle$ is virtually cyclic and contains a loxodromic element.

(iii) There exists an α-reduced subset $S \subset U^c$ such that

$$|S| \geqslant \max\left\{2, \frac{1}{c}|U|\right\}.$$

Moreover,

$$\omega(U) \geqslant \frac{1}{c}\log|U|.$$

Our main contribution to Theorem 2.3.8 is the dependence of the involved constants: for our purpose it is important that the number c only depends on the acylindricity parameters κ and N.

Remark 0.6.11. — If the injectivity radius of the action of G on X is large, then every finite symmetric subset of G satisfies either (ii) or (iii). In general this is however not the case. We will later use uniform uniform exponential growth of G in order to apply Theorem 2.3.8 to some power of an arbitrary symmetric subset U in G.

Theorem 2.3.8 with *Fekete's Subadditive Lemma* and the fact that $\omega(U^n) = n\omega(U)$ implies the following corollary. It is a weak form of *purely exponential growth*, [25, 78].

Corollary 0.6.12. — *For every κ, $N > 0$, there exists $\xi > 1$ with the following property. Let $\delta > 0$ and $\kappa \geqslant \delta$. Let G be a group acting (κ, N)-acylindrically on a δ-hyperbolic space X. Let $U \subset G$ be a finite symmetric subset containing the identity of energy $\mathrm{L}(U) > 10^4\kappa$ that does not generate a virtually cyclic subgroup. Then, for every $n \geqslant 0$,*

$$e^{n\omega(U)} \leqslant |U^n| \leqslant e^{\xi n\omega(U)}.$$

To prove Theorem 0.6.2 (i), we follow a strategy of [36] that estimates product set growth in Burnside groups. In particular, we use the viewpoint of geometric small cancellation theory. As previously mentioned, the *Small Cancellation Theorem* gives a universal constant $\bar{\delta} > 0$ such that any geometric small cancellation quotient $\bar{G}$ of a group G acting acylindrically on a δ-hyperbolic space X, for appropriate choice of the small cancellation parameters, acts acylindrically on a $\bar{\delta}$-hyperbolic space $\bar{X}$. Let $\bar{U} \subset \bar{G}$ be a finite symmetric generating set containing the identity that is not contained in an elliptic or virtually cyclic subgroup. If the energy of $\bar{U}$ is larger than $10^4\bar{\delta}$, then the exponential growth rate of $\bar{U}$ is bounded below by a universal strictly positive constant (Lemma 2.1.23). Otherwise, we fix a pre-image U of $\bar{U}$ in G of minimal energy for the action of G on X (Lemma 2.1.32). Such a pre-image may not have large energy $> 10^4\delta$. Indeed, it may consist entirely of torsion-elements and thus have small energy $< 10^4\delta$. However, our pre-image U is not contained in any elliptic subgroup. Thus some power of U contains a loxodromic element, hence, for some exponent n, we have $\mathrm{L}(U^n) > 10^4\delta$. We stress that the exponent n depends on the set U. We now apply Theorem 0.6.10 to U^n. Since U is not contained in any virtually cyclic subgroup, we obtain a reduced subset S in U^{cn}, which freely generates a free subgroup. Next, we adapt the counting argument of [29, 36] to prove that for every $r \geqslant 1$, the proportion of elements in S^r that contain a large part of a relator is small compared to $|S^r|$ (Proposition 2.4.9). A combination of a consequence of *Greendlinger's Lemma* (Proposition 2.4.16) and *Fekete's Subadditive Lemma* then implies that the exponential growth rate of $\bar{U}$ satisfies

$$\omega(\bar{U}) \geqslant \beta \cdot \omega(U).$$

for

$$\beta = \sup_{\theta\in(0,1)} \inf \left\{ \theta \cdot \frac{\log \frac{3}{2}}{\log(2c)}, 1-\theta \right\} \cdot \frac{1}{c}.$$

Finally, assume that G has ξ-uniform uniform exponential growth. A combination of this fact with the previous inequality yields Theorem 0.6.2 (i). The proof of Theorem 0.6.2 (ii) is similar and we postpone its discussion.

NOTATION

Let X be a metric space. Given two points $x, x' \in X$, we write $|x - x'|$ for the distance between them. The *ball of* X of center $x \in X$ and radius $r > 0$ is

$$B_X(x,r) = \{\, y \in X \ : \ |x - y| \leqslant r \,\}.$$

The *distance between a point* $x \in X$ *and a subset* $Y \subset X$ is

$$d(x,Y) = \inf \{\, |x - y| \ : \ y \in Y \,\}.$$

Let $\eta \geqslant 0$. The η*-neighbourhood* of a subset $Y \subset X$ is

$$Y^{+\eta} = \{\, x \in X \ : \ d(x,Y) \leqslant \eta \,\}.$$

The *distance* between two subsets $Y, Z \subset X$ is

$$d(Y,Z) = \inf \{\, |y - z| \ : \ y \in Y, z \in Z \,\}.$$

The *Hausdorff distance* between two subsets $Y, Z \subset X$ is

$$d_{\mathrm{Haus}}(Y,Z) = \inf \{\, \varepsilon \geqslant 0 \ : \ Y \subset Z^{+\varepsilon} \text{ and } Z \subset Y^{+\varepsilon} \,\}.$$

A *path* is a continuous map $\alpha\colon [a,b] \to X$. The *initial and terminal points* of α are $\alpha(a)$ and $\alpha(b)$, respectively. We denote by α^- and α^+ the initial and terminal points of α, respectively. They form the *endpoints* of α. We will frequently identify a path and its image. A *subpath* of α is a restriction of α to a subinterval of $[a,b]$. The path α *joins* the point $x \in X$ to the point $y \in X$ if $\alpha^- = x$ and $\alpha^+ = y$. Note that for every $x, y \in \alpha$ there may be more than one subpath of α joining x to y, unless the points are given by the parametrisation of α. If $x, y \in \alpha$ are given by the parametrisation, we denote by $[x,y]_\alpha$ the parametrised subpath of α joining x to y. The *length* of a path α is denoted by $\ell(\alpha)$. If α joins a point $x \in X$ to a point $y \in Y$ of a closed subset $Y \subset X$, the *entrance point* of

α in Y is the point $y' \in \alpha$ satisfying

$$\ell([x,y']_\alpha) = \inf_{z \in \alpha \cap Y} \ell([x,z]_\alpha).$$

Unless otherwise stated a path is a *rectifiable path* parametrised by *arc length*. Let $\kappa \geqslant 1$, $l \geqslant 0$. A path $\alpha\colon [a,b] \to X$ is a (κ, l)-*quasi-geodesic* if for every $t, t' \in [a,b]$,

$$|\alpha(t) - \alpha(t')| \leqslant |t - t'| \leqslant \kappa|\alpha(t) - \alpha(t')| + l.$$

Note that that $\ell(\alpha_{|[t,t']}) = |t - t'|$. Let $L \geqslant 0$. We say that α is a L-*local* (κ, l)-*quasi-geodesic* if any subpath of α whose length is at most L is a (κ, l)-quasi-geodesic. A *geodesic* is a $(1,0)$-quasi-geodesic. The metric space X is *geodesic* if for every pair of points $x, x' \in X$ there exists a geodesic of X joining x to x'. We write $[x, x']$ for a geodesic joining them. Recall that there may be multiple geodesics joining two points.

Chapter 1

Growth of quasi-convex subgroups in groups with a constricting element

THE GIANT: The first thing I will tell you is: *"There's a man in a smiling bag"*. The second thing is: *"The owls are not what they seem"*. The third thing is: *"Without chemicals, he points"*.
COOPER: What do these things mean?
THE GIANT: This is all I am permitted to say.

from *Twin Peaks*, created by David Lynch and Mark Frost

Contents

The results of this chapter correspond to the following article:

- X. Legaspi. Constricting elements and the growth of quasi-convex subgroups, 2022. URL: https://orcid.org/0000-0002-1497-6448.

In Section 1.1 we will introduce the definitions of path system group, quasi-convex subgroup and constricting element. We also state some standard properties (Proposition A.1.1) that will be proven in the Appendix A. In Section 1.2 we will explain the two criteria that we will use to estimate the growth of quasi-convex subgroups. The rest of the chapter is devoted to the development of our geometric framework so that we can apply these criteria. In Section 1.3 we will prove a version of the Bounded Geodesic Image Property of hyperbolic spaces, but for quasi-convex subsets insetad of geodesics. In Section 1.4 we will introduce the notion of *buffering sequence* and we will give a version of Behrstock's inequality. In Section 1.5, given an infinite index quasi-convex subgroup and a quasi-convex element, we will produce another quasi-convex element whose orbit is "transversal" to the given subgroup. The proofs of both of our main results (Theorem 0.5.8 and Theorem 0.5.13) share this argument. In Section 1.6 we will study the elementary closures of constricting elements and also some geometric separation properties. Finally, in Section 1.7 we will prove our main results (including Theorem 0.5.10) by constructing an appropriate buffering sequence in each situation.

1.1 Path system geometry

This section is devoted to present the notations and vocabulary of the main geometric objects of this chapter. We formalise our notions of "convex cocompactness" and "non-positive curvature".

Path system spaces.

Definition 1.1.1 (Path system space). — Let $\mu \geqslant 1$, $\nu \geqslant 0$. A (μ,ν)-*path system space* $(X, \mathscr{P})$ is a metric space X together with a collection $\mathscr{P}$ of paths of X satisfying:

(PS1) $\mathscr{P}$ is closed under taking subpaths.

(PS2) For every $x, y \in X$, there exists $\gamma \in \mathscr{P}$ joining x to y.

(PS3) Every element of $\mathscr{P}$ is a (μ,ν)-quasi-geodesic.

We refer to $\mathscr{P}$ as (μ,ν)-*path system.*

We fix $\mu \geqslant 1$, $\nu \geqslant 0$ and a (μ,ν)-*path system space* $(X, \mathscr{P})$.

Definition 1.1.2 (Quasi-convex subset). — Let $\eta \geqslant 0$. A *subset* $Y \subset X$ is η-*quasi-convex* if every $\gamma \in \mathscr{P}$ with endpoints in Y is contained in the η-neighbourhood of Y.

Definition 1.1.3 (Constricting subset). — Let $\delta \geqslant 0$. A *subset* $A \subset X$ is δ-*constricting* if there exists a map $\pi_A \colon X \to A$ satisfying:

(CS1) **Coarse retraction.**
For every $x \in A$, we have $|x - \pi_A(x)| \leqslant \delta$.

(CS2) **Constriction.**
For every $x, y \in X$ and for every $\gamma \in \mathscr{P}$ joining x to y, if we have $|\pi_A(x) - \pi_A(y)| > \delta$, then $\gamma \cap B_X(\pi_A(x), \delta) \neq \varnothing$ and $\gamma \cap B_X(\pi_A(y), \delta) \neq \varnothing$.

We refer to $\pi_A \colon X \to A$ as δ-*constricting map.*

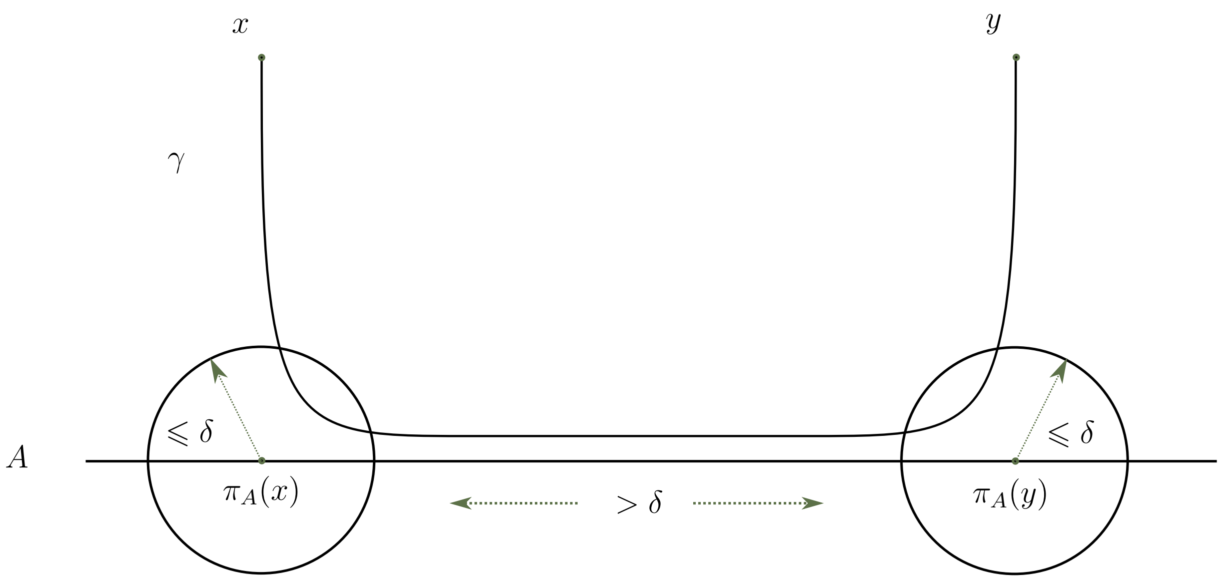

Figure 1.1 – The constriction property.

Notation 1.1.4. — Let $\pi_A \colon X \to A$ be a map between X and a subset $A \subset X$. For every $x, y \in X$, we denote $|x - y|_A = |\pi_A(x) - \pi_A(y)|$. For every subset $Y \subset X$, we denote $\operatorname{diam}_A(Y) = \operatorname{diam}(\pi_A(Y))$. For every $x \in X$ and for every pair of subsets $Y, Z \subset X$, we denote $d_A(x, Y) = d(\pi_A(x), \pi_A(Y))$ and $d_A(Y, Z) = d(\pi_A(Y), \pi_A(Z))$. Note that d_A may not be a distance over the collection of subsets of X: it may not satisfy the triangle inequality. We will keep this notation for the rest of the paper.

The following are some standard properties:

Proposition 1.1.5. — *For every $\delta \geqslant 0$, there exist a constant $\theta \geqslant 0$ and a pair of maps, $\sigma \colon \mathbf{R}_{\geqslant 1} \times \mathbf{R}_{\geqslant 0} \to \mathbf{R}_{\geqslant 0}$ and $\zeta \colon \mathbf{R}_{\geqslant 0} \to \mathbf{R}_{\geqslant 0}$, such that any δ-constricting map $\pi_A \colon X \to A$ satisfies the following properties:*

(1) ***Coarse nearest-point projection.***

For every $x \in X$, we have $|x - \pi_A(x)| \leqslant \mu d(x, A) + \theta$.

(2) ***Coarse equivariance.***

Let H be a group acting by isometries on X such that A and $\mathscr{P}$ are H-invariant. Then for every $h \in H$ and for every $x \in X$, we have $|\pi_A(hx) - h\pi_A(x)| \leqslant \theta$.

(3) ***Coarse Lipschitz map.***

For every $x, y \in X$, we have $|x - y|_A \leqslant \mu|x - y| + \theta$.

(4) ***Intersection–Image.***

For every $\gamma \in \mathscr{P}$, we have $|\operatorname{diam}(A^{+\delta} \cap \gamma) - \operatorname{diam}_A(\gamma)| \leqslant \theta$.

(5) ***Behrstock inequality.***

Let $\pi_B \colon X \to B$ be a δ-constricting map. Then for every $x \in X$, we have

$$\min\{d_A(x, B), d_B(x, A)\} \leqslant \theta.$$

(6) ***Morseness.***

Let $\kappa \geqslant 1$, $l \geqslant 0$. Let α be a (κ, l)-quasi-geodesic of X with endpoints in A. Then $\alpha \subset A^{+\sigma(\kappa,l)}$.

(7) ***Coarse invariance.***

Let $\varepsilon \geqslant 0$. Let $B \subset X$ be a subset such that $d_{\text{Haus}}(A, B) \leqslant \varepsilon$. Then B is $\zeta(\varepsilon)$-constricting.

Proof. — We give some references. For (1), (3) and (4), see [74, Lemma 2.4]. For (5), see [74, Lemma 2.5]. For (6), see [74, Lemma 2.8 (1)]. We leave the proof of the properties (2) and (7) as an exercise. □

Path system groups. Let G be a group acting by isometries on a metric space X. The *quasi-stabilizer* $\operatorname{Stab}_G(x, r)$ of $x \in X$ of radius $r \geqslant 0$ is defined as

$$\operatorname{Stab}_G(x, r) = \{g \in G \colon |x - gx| \leqslant r\}.$$

The action of G on X is *proper* if for every $x \in X$ and for every $r \geqslant 0$, we have $|\operatorname{Stab}_G(x, r)| < \infty$. Let $\eta \geqslant 0$. The action of G on X is *η-cobounded* if for every $x, x' \in X$, there exists $g \in G$ such that $|x - gx'| \leqslant \eta$.

Definition 1.1.6 (Path system group). — Let $\mu \geqslant 1$, $\nu \geqslant 0$. A (μ,ν)-*path system group* $(G, X, \mathscr{P})$ is a group G acting properly on a metric space X together with a G-invariant collection $\mathscr{P}$ of paths of X such that $(X, \mathscr{P})$ is a (μ,ν)-path system space.

We fix $\mu \geqslant 1$, $\nu \geqslant 0$ and a (μ,ν)-*path system group* $(G, X, \mathscr{P})$.

Definition 1.1.7 (Quasi-convex subgroup). — A *subgroup* $H \leqslant G$ is η-*quasi-convex* if there exists an H-invariant η-quasi-convex subset $Y \subset X$ such that the action of H on Y is η-cobounded. We will write (H, Y) when we need to stress the η-quasi-convex subset Y that H is preserving.

Definition 1.1.8 (Constricting element). — Let $\delta \geqslant 0$. An *element* $g \in G$ is δ-*constricting* if the following holds:

(CE1) g has infinite order.

(CE2) There exists a $\langle g\rangle$-invariant δ-constricting subset $A \subset X$ so that the action of $\langle g\rangle$ on A is δ-cobounded.

We will write (g, A) when we need to stress the δ-constricting subset A that $\langle g\rangle$ is preserving.

1.2 Growth estimation criteria

In this section, we fix a group G acting properly on a metric space X and a subgroup $H \leqslant G$. The goal is to establish simple criteria so that we can check if H is a solution to the system of equations

$$\begin{cases} \omega(H) < \omega(G), \\ \omega(G/H) = \omega(G). \end{cases}$$

Our criterion to estimate the relative exponential growth rate is basically [39, Criterion 2.4]. The statement that we actually need is more specific, so we will give a proof for the convenience of the reader. Recall that the action of a subgroup $H \leqslant G$ on X is *divergent* if its Poincaré series $\mathscr{P}_H(s)$ diverges at $s = \omega(H)$.

Proposition 1.2.1 ([39, Criterion 2.4]). — *Assume that the following conditions are true:*

(i) $\omega(H) < \infty$.

(ii) The action of H *on* X *is divergent.*

(iii) *There exist subgroups $K \leqslant G$ and $F \leqslant H \cap K$ so that F is a proper finite subgroup of K and the natural homomorphism $\phi\colon H *_F K \to G$ is injective.*

Then $\omega(H) < \omega(G)$.

Remark 1.2.2. — In the proof below, note that the relative exponential growth rate makes sense for any subset of G, as it does the notion of Poincaré series.

Proof. — Since the action of H on X is divergent, in particular H is infinite and hence $H - F$ is non-empty. Since F is a proper subgroup of K, there exists $k \in K - F$. Denote by U the set of elements of $H *_F K$ that can be written as words that alternate elements of $H - F$ and k, always with an element of $H - F$ at the beginning and with a k at the end. The inequality $\omega(\phi(U)) \leqslant \omega(G)$ can be deduced from the definition. It is enough to prove that there exists $s_0 \geqslant 0$ such that $\omega(H) < s_0 \leqslant \omega(\phi(U))$. Let $o \in X$. Since $\omega(H) < \infty$, the interval $(\omega(H), \infty)$ is non-empty. Since the action of H on X is divergent, there exists $s_0 \in (\omega(H), \infty)$ such that $\sum_{h\in H-F} e^{-s_0|o-hko|} > 1$; otherwise one obtains a contradiction with the divergence of the action of H on X.

In order to obtain the inequality $s_0 \leqslant \omega(\phi(U))$, it suffices to show that the Poincaré series $\mathscr{P}_{\phi(U)}(s) = \sum_{g\in\phi(U)} e^{-s|o-go|}$ diverges at $s = s_0$. Since $\phi\colon H *_F K \to G$ is injective, we have

$$\mathscr{P}_{\phi(U)}(s) \geqslant \sum_{m\geqslant 1} \sum_{h_1,\cdots,h_m\in H-F} e^{-s|o-h_1kh_2k\cdots h_mko|}.$$

By the triangle inequality, for every $m \geqslant 1$ and for every $h_1, \cdots, h_m \in H - F$, we have $|o - h_1kh_2k\cdots h_mko| \leqslant \sum_{i=1}^{m} |o - h_iko|$. Thus,

$$\sum_{h_1,\cdots,h_m\in H-F} e^{-s|o-h_1kh_2k\cdots h_mko|} \geqslant \left[\sum_{h\in H-F} e^{-s|o-hko|}\right]^m.$$

We see that $\mathscr{P}_H(s_0) = \infty$ follows from the claim. □

Our criterion to estimate the quotient exponential growth rate is the following:

Definition 1.2.3. — Let $\phi\colon G \to G$. We say that G is *ϕ-coarsely G/H* if there exist $\theta \geqslant 0$ and $x \in X$ satisfying the following conditions:

(CQ1) For every $u, v \in G$, if $\phi(u)H = \phi(v)H$, then $|\phi(u)x - \phi(v)x| \leqslant \theta$.

(CQ2) For every $u \in G$, $|ux - \phi(u)x| \leqslant \theta$.

Proposition 1.2.4. — *If there exist $\phi\colon G \to G$ such that G is ϕ-coarsely G/H, then $\omega(G) = \omega(G/H)$.*

Proof. — The inequality $\omega(G/H) \leqslant \omega(G)$ can be deduced from the defintion. Assume that there exist $\phi\colon G \to G$ such that G is ϕ-coarsely G/H for $x \in X$ and $\theta \geqslant 0$.

Claim 1.2.5. — There exist $\kappa \geqslant 1$ such that for every $r > 0$,

$$|\operatorname{Stab}_G(x,r)| \leqslant \kappa |p(\operatorname{Stab}_G(x, r+\theta))|.$$

Let $\kappa = |\operatorname{Stab}_G(x, 3\theta)|$. Let $r > 0$. Let $p\colon G \twoheadrightarrow G/H$ be the natural projection. Let $q\colon G \to G/H$ the map that sends u to $\phi(u)H$. Note that the quasi-stabilizer $\operatorname{Stab}_G(x,r)$ can be decomposed as the disjoint union of the sets $q^{-1}(q(u))$ such that $q(u) \in q(\operatorname{Stab}_G(x,r))$. Hence,

$$|\operatorname{Stab}_G(x,r)| \leqslant \sum_{q(u) \in q(\operatorname{Stab}_G(x,r))} |q^{-1}(q(u))|.$$

It suffices to estimate the size of $q(\operatorname{Stab}_G(x,r))$ and the size of $q^{-1}(q(u))$, for every $u \in G$. First we prove that $|q(\operatorname{Stab}_G(x,r))| \leqslant |p(\operatorname{Stab}_G(x, r+\theta))|$. Let $u \in \operatorname{Stab}_G(x,r)$. By the triangle inequality,

$$|x - \phi(u)x| \leqslant |x - ux| + |ux - \phi(u)x|.$$

By the hypothesis (CQ2), we have $|ux - \phi(u)x| \leqslant \theta$. Hence $|x - \phi(u)x| \leqslant r + \theta$. Consequently, $q(\operatorname{Stab}_G(x,r)) \subset p(\operatorname{Stab}_G(x, r+\theta))$. Now we prove that for every $u \in G$, we have $|q^{-1}(q(u))| \leqslant \kappa$. Let $u \in G$. Since $|u \operatorname{Stab}_G(x, 3\theta)| = |\operatorname{Stab}_G(x, 3\theta)| = \kappa$, it is enough to prove that $u^{-1}q^{-1}(q(u)) \subset \operatorname{Stab}_G(x, 3\theta)$. Let $v \in q^{-1}(q(u))$. By the triangle inequality,

$$|x - u^{-1}vx| = |ux - vx| \leqslant |ux - \phi(u)x| + |\phi(u)x - \phi(v)x| + |\phi(v)x - vx|.$$

Since $q(u) = q(v)$, we have that $\phi(u)H = \phi(v)H$. It follows from the hypothesis (CQ1) that $|\phi(u)x - \phi(v)x| \leqslant \theta$. By the hypothesis (CQ2), we have $\max\{|ux - \phi(u)x|, |vx - \phi(v)x|\} \leqslant \theta$. Thus, $|x - u^{-1}vx| \leqslant 3\theta$. This proves the claim.

Consequently,

$$\omega(G) \leqslant \limsup_{r\to\infty} \frac{1}{r} \log |p(\operatorname{Stab}_G(x, r+\theta))|.$$

Finally, observe that

$$\limsup_{r\to\infty} \frac{1}{r} \log |p(\operatorname{Stab}_G(x, r+\theta))| = \limsup_{r\to\infty} \frac{r+\theta}{r} \frac{1}{r+\theta} \log |p(\operatorname{Stab}_G(x, r+\theta))|.$$

Hence $\omega(G) \leqslant \omega(G/H)$. □

1.3 Buffering sequences

In this section, we fix constants $\mu \geqslant 1$, $\nu \geqslant 0$ and a (μ,ν)-path system space $(X, \mathscr{P})$. Despite the fact that our space X does not carry any global geometric condition, we still can obtain some control through constricting subsets. We could ignore the "wild regions" if, for instance, we were able to "jump" from one constricting subset to another. The buffering sequences below encapsulate this idea. In fact, the proofs of our main results consist essentially in building up some particular buffering sequences. W. Yang had already introduced this concept for piece-wise geodesics in [78].

Definition 1.3.1. — Let δ, ε, $L \geqslant 0$. Let $\mathscr{A}$ be a collection of subsets of X. A finite sequence of subsets $Y_0, A_1, Y_1, \cdots, A_n, Y_n \subset X$ where Y_0 and Y_n are the only possible empty sets is (δ,ε,L)-*buffering on* $\mathscr{A}$ if for every $i \in [\![1,n]\!]$ the set A_i belongs to $\mathscr{A}$ and there exists a δ-constricting map $\pi_{A_i} \colon X \to A_i$ with the following properties whenever Y_i and Y_{i-1} are non-empty:

(BS1) $\max\{\operatorname{diam}_{A_i}(A_{i+1}), \operatorname{diam}_{A_{i+1}}(A_i)\} \leqslant \varepsilon$ if $i \neq n$.

(BS2) $\max\{\operatorname{diam}_{A_i}(Y_{i-1}), \operatorname{diam}_{A_i}(Y_i)\} \leqslant \varepsilon$.

(BS3) $\max\{d(A_i, Y_{i-1}), d(A_i, Y_i)\} \leqslant \varepsilon$.

(BS4) $d_{A_i}(Y_{i-1}, Y_i) \geqslant L$.

What makes buffering sequences remarkable is that they satisfy a variant of *Behrstock inequality*. We will find a direct application of the following inequality later in the study of the quotient exponential growth rates:

Proposition 1.3.2. — *For every δ, $\varepsilon \geqslant 0$, there exists $\theta \geqslant 0$ with the following property. Let $A, Y, B \subset X$ be a $(\delta,\varepsilon,0)$-buffering sequence on $\{A, B\}$. Then for every $x \in X$,*

$$\min\{d_A(x,Y), d_B(x,Y)\} \leqslant \theta.$$

Proof. — Let δ, $\varepsilon \geqslant 0$. Let $\theta_0 = \theta_0(\delta) \geqslant 0$ be the constant of Proposition 1.1.5. Let $\theta > \theta_0 + 1$. Its exact value will be precised below. Let $A, Y, B \subset X$ be a $(\delta,\varepsilon,0)$-buffering sequence on $\{A, B\}$. Let $x \in X$. By symmetry, it suffices to show that if $d_A(x,Y) > \theta$, then $d_B(x,Y) \leqslant \theta$. Assume that $d_A(x,Y) > \theta$. Let $a \in A$ such that $|x-a|_B \leqslant d_B(x,A)+1$. Let $b \in B$. Let $y \in Y$. By (BS3), we have $\max\{d(A,Y), d(B,Y)\} \leqslant \varepsilon$; hence there exist $p \in A^{+\varepsilon+1} \cap Y$ and $q \in B^{+\varepsilon+1} \cap Y$. It follows from the definition of buffering sequence that

$$\max\{|b-\pi_B(q)|_A, |q-p|_A, |a-\pi_A(p)|_B, |p-y|_B\} \leqslant \varepsilon.$$

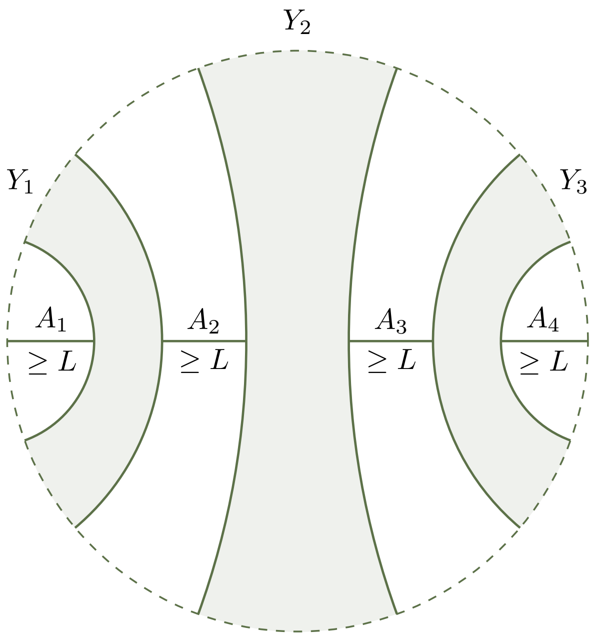

Figure 1.2 – An example of a buffering sequence in the Poincaré disk model. In this example, the sets A_i are subpaths of length $\geqslant L$ of a given bi-infinite geodesic α. Each set Y_i is the collection of geodesics that are orthogonal to the geodesic segment of α that is between A_i and A_{i+1}. In particular, the sets Y_i are quasi-convex. For more intuition, one could interpret this picture on a tree.

Applying together Proposition 1.1.5 (1) *Coarse nearest-point projection* and (3) *Coarse Lipschitz map*, we obtain

$$\max\{|\pi_B(q) - q|_A, |\pi_A(p) - p|_B\} \leqslant \mu^2(\varepsilon + 1) + \mu\theta_0 + \theta_0.$$

Claim 1.3.3. — $d_A(x, B) > \theta_0$

By the triangle inequality,

$$|x - b|_A \geqslant |x - p|_A - |b - \pi_B(q)|_A - |\pi_B(q) - q|_A - |q - p|_A.$$

Moreover, $|x - p|_A \geqslant d_A(x, Y)$. Since the element b is arbitrary and we have $d_A(x, Y) > \theta_0 + 1$, we obtain $d_A(x, B) > \theta_0$. This proves the claim.

Finally, we are going to estimate $d_B(x, Y)$. By the triangle inequality,

$$|x - y|_B \leqslant |x - a|_B + |a - \pi_A(p)|_B + |\pi_A(p) - p|_B + |p - y|_B.$$

Since $d_A(x, B) > \theta_0$, it follows from Proposition 1.1.5 (5) *Behrstock inequality* and the definition of a that $|x-a|_B \leqslant \theta_0+1$. Since the element y is arbitrary, we obtain $d_B(x, Y) \leqslant \theta$ for $\theta = 2\theta_0 + 1 + 2\varepsilon + \mu^2(\varepsilon + 1) + \mu\theta_0$. □

The corollary below will be applied to the study of the relative exponential growth rates:

Corollary 1.3.4. — *For every δ, ε, $\theta \geqslant 0$ there exists $L \geqslant 0$ with the following property. Let $Y_0, A_1, Y_1, \cdots, A_n, Y_n \subset X$ be an (δ, ε, L)-buffering sequence on $\{A_i\}$. Then for every $i \in [\![1, n]\!]$,*

$$d_{A_i}(Y_0, Y_i) > \theta.$$

Proof. — Let δ, ε, $\theta \geqslant 0$. Let $\theta_0 = \theta_0(\delta, \varepsilon) \geqslant 0$ be the constant of Proposition 1.3.2. We put $L = \theta + \theta_0 + 1$. Let $y_0 \in Y_0$. Let $i \in [\![1, n]\!]$.

Claim 1.3.5. — $d_{A_i}(y_0, Y_i) \geqslant d_{A_i}(Y_{i-1}, Y_i) - d_{A_i}(y_0, Y_{i-1})$.

Let $y_{i-1} \in Y_{i-1}$ and $y_i \in Y_i$. By the triangle inequality,

$$|y_0 - y_i|_{A_i} \geqslant |y_{i-1} - y_i|_{A_i} - |y_0 - y_{i-1}|_{A_i}.$$

Note that $|y_{i-1} - y_i|_{A_i} \geqslant d_{A_i}(Y_{i-1}, Y_i)$. Since the elements y_{i-1}, y_i are arbitrary, this proves the claim.

Finally, we prove by induction on $i \in [\![1, n]\!]$ that, $d_{A_i}(Y_0, Y_i) > \theta$. If $i = 1$, then $d_{A_1}(Y_0, Y_1) > \theta$ follows from (BS4), since $L > \theta$. Assume that $i \in [\![1, n-1]\!]$ and $d_{A_i}(Y_0, Y_i) > \theta$. Then $d_{A_i}(y_0, Y_i) > \theta_0$. It follows from Proposition 1.3.2 that $d_{A_{i+1}}(y_0, Y_i) \leqslant \theta_0$. By (BS4), $d_{A_{i+1}}(Y_i, Y_{i+1}) \geqslant L$. Applying the previous claim, we obtain $d_{A_{i+1}}(y_0, Y_{i+1}) > \theta$. Since the element y_0 is arbitrary, $d_{A_{i+1}}(Y_0, Y_{i+1}) > \theta$. This concludes the inductive step. □

1.4 Quasi-convexity on Intersection–Image property

In this section, we fix constants $\mu \geqslant 1$, $\nu \geqslant 0$ and a (μ, ν)-path system space $(X, \mathscr{P})$. In this section, we prove a variant of Proposition 1.1.5 (4) *Intersection–Image*. Basically, we will be exchanging paths of $\mathscr{P}$ for quasi-convex subsets of X, further thickening the involved sets.

Proposition 1.4.1. — *For every δ, $\eta \geqslant 0$, there exist $\theta \geqslant 0$ and $\zeta\colon \mathbf{R}_{\geqslant 0} \times \mathbf{R}_{\geqslant 0} \to \mathbf{R}_{\geqslant 0}$ with the following property. Let $\pi_A\colon X \to A$ be a δ-constricting map. Let Y be an η-quasi-convex subset of X. Let $\varepsilon_1 \geqslant 0$, $\varepsilon_2 \geqslant 0$. Then*

$$|\operatorname{diam}(A^{+\theta+\varepsilon_1} \cap Y^{+\varepsilon_2}) - \operatorname{diam}_A(Y)| \leqslant \zeta(\varepsilon_1, \varepsilon_2).$$

Proof. — Let δ, $\eta \geqslant 0$. Let $\theta_0 = \theta_0(\delta) \geqslant 0$ be the constant of Proposition 1.1.5. We put $\theta = \delta + \eta + 1$. Let $\zeta\colon \mathbf{R}_{\geqslant 0} \times \mathbf{R}_{\geqslant 0} \to \mathbf{R}_{\geqslant 0}$ depending on δ, η. Its exact value will be precised below. Let $\pi_A\colon X \to A$ be a δ-constricting map. Let Y be an η-quasi-convex subset of X. Let $\varepsilon_1 \geqslant 0$, $\varepsilon_2 \geqslant 0$.

First we prove that $\operatorname{diam}_A(Y) \leqslant \operatorname{diam}(A^{+\theta+\varepsilon_1} \cap Y^{+\varepsilon_2}) + \zeta(\varepsilon_1, \varepsilon_2)$. Let $x, y \in Y$. It suffices to assume that $|x - y|_A > \delta$. Let $\gamma \in \mathscr{P}$ joining x to y. By (CS2), there exist $p, q \in \gamma$ such that

$$\max\{|\pi_A(x) - p|, |\pi_A(y) - q|\} \leqslant \delta.$$

Since the subset Y is η-quasi-convex, there exist $p', q' \in Y$ such that

$$\max\{|p - p'|, |q - q'|\} \leqslant \eta + 1.$$

By the triangle inequality,

$$|x - y|_A \leqslant |\pi_A(x) - p| + |p - p'| + |p' - q'| + |q' - q| + |q - \pi_A(y)|.$$

Since $p', q' \in A^{+\theta+\varepsilon_1} \cap Y^{+\varepsilon_2}$, we have $|p'-q'| \leqslant \operatorname{diam}(A^{+\theta+\varepsilon_1} \cap Y^{+\varepsilon_2})$. Hence,

$$|x-y|_A \leqslant \operatorname{diam}(A^{+\theta+\varepsilon_1} \cap Y^{+\varepsilon_2}) + 2\delta + 2\eta + 1.$$

Now we prove that $\operatorname{diam}(A^{+\theta+\varepsilon_1} \cap Y^{+\varepsilon_2}) \leqslant \operatorname{diam}_A(Y) + \zeta(\varepsilon_1, \varepsilon_2)$. Let $x, y \in A^{+\theta+\varepsilon_1} \cap Y^{+\varepsilon_2}$. Since $x, y \in Y^{+\varepsilon_2}$, there exist $x', y' \in Y$ such that $\max\{|x-x'|, |y-y'|\} \leqslant \varepsilon_2 + 1$. By the triangle inequality,

$$|x-y| \leqslant |x-\pi_A(x)| + |x-x'| + |x'-y'|_A + |y'-y|_A + |\pi_A(y)-y|.$$

Since $x, y \in A^{+\theta+\varepsilon_1}$, it follows from Proposition 1.1.5 (1) *Coarse nearest-point projection* that

$$\max\{|x-\pi_A(x)|, |y-\pi_A(y)|\} \leqslant \mu(\theta+\varepsilon_1) + \theta_0.$$

It follows from Proposition 1.1.5 (3) *Coarse Lipschitz Map* that,

$$\max\{|x-x'|_A, |y-y'|_A\} \leqslant \mu(\varepsilon_2+1) + \theta_0.$$

Since $\pi_A(x'), \pi_A(y') \in \pi_A(Y)$, we have $|x'-y'|_A \leqslant \operatorname{diam}_A(Y)$. Hence,

$$|x-y| \leqslant \operatorname{diam}_A(Y) + 2\mu(\theta+\varepsilon_1) + 2\mu(\varepsilon_2+1) + 4\theta_0.$$

Finally, we put $\zeta(\varepsilon_1, \varepsilon_2) = \max\{2\delta + 2\eta + 1, 2\mu(\theta+\varepsilon_1) + 2\mu(\varepsilon_2+1) + 4\theta_0\}$. □

Applying the symmetry of Proposition 1.4.1 in combination with Proposition 1.1.5 (6) *Morseness* and (7) *Coarse invariance*, we deduce:

Corollary 1.4.2. — *For every $\delta \geqslant 0$, there exists $\theta \geqslant 0$ with the following property. Let $\pi_A \colon X \to A$ and $\pi_B \colon X \to B$ be δ-constricting maps. Then:*

$$|\operatorname{diam}_A(B) - \operatorname{diam}_B(A)| \leqslant \theta.$$

1.5 Finding a quasi-convex element

Given a torsion-free hyperbolic group G containing a loxodromic element g_0 and an infinite index quasi-convex subgroup H, one can find another loxodromic element $g \in G$ conjugate to g_0 so that H has trivial intersection with $\langle g \rangle$ [6, Theorem 1]. The goal of this section is to reimplement this fact in our setting, using a "quasi-convex element" instead of a loxodromic element. In this section, we fix constants $\mu \geqslant 1$, $\nu \geqslant 0$ and a (μ, ν)-path system group $(G, X, \mathscr{P})$.

Definition 1.5.1 (Quasi-convex element). — Let $\eta \geqslant 0$. An *element* $g \in G$ is *η-quasi-convex* if the following holds:

(QE1) g has infinite order.

(QE2) $\langle g \rangle$ is an η-quasi-convex subgroup of G.

We will write (g, A) when we need to stress the η-quasi-convex subset A that $\langle g \rangle$ is preserving.

The main result of this section is the following.

Proposition 1.5.2. — *Let $\eta \geqslant 0$. Assume that G contains an η-quasi-convex element (g, A). There exists $\theta = \theta(\eta, g, A) \geqslant 1$ satisfying the following. Let (H, Y) be an η-quasi-convex subgroup of G. Then:*

(i) For every $u \in G$, if $\operatorname{diam}(uA \cap Y) > \theta$, then $uA \subset Y^{+\theta}$.

(ii) Let $H \leqslant K \leqslant G$. If $[K : H] > \theta$, then there exist $k \in K$ such that $\operatorname{diam}(kA \cap Y) \leqslant \theta$.

Remark 1.5.3. — Under the notation of (ii), when $K = G$, the element kgk^{-1} has the desired property that we were looking for. Note that (kgk^{-1}, kA) is quasi-convex since $\mathscr{P}$ is G-invariant.

The rest of the section is devoted to the proof of Proposition 1.5.2.

Definition 1.5.4. — Let $\kappa \geqslant 1$, $l \geqslant 0$. A map $\phi\colon (Y, d_Y) \to (Z, d_Z)$ between two metric spaces is a (κ, l)-*quasi-isometric embedding* if for every $y, y' \in Y$,

$$\frac{1}{\kappa} d_Y(y, y') - l \leqslant d_Z(\phi(y), \phi(y')) \leqslant \kappa d_Y(y, y') + l.$$

We start with a variant of *Milnor-Schwarz Theorem*. If U is a generating set of a group H, we denote by d_U the word metric of H with respect to U.

Lemma 1.5.5. — *For every $\eta \geqslant 0$, there exist $\theta \geqslant 1$ with the following property. Let (H, Y) be an η-quasi-convex subgroup of G. For every $y \in Y$, there exists a finite generating set U of H such that the orbit map $(H, d_U) \to X$, $h \mapsto hy$ is a (θ, θ)-quasi-isometric embedding.*

For the proof, one can use the same kind of argument as that of *Milnor-Schwarz Theorem*, but bearing in mind that Y might not be a length metric space, which is required by the original statement. The only difference here is that one uses the paths of $\mathscr{P}$ with endpoints in Y. They are enough for the proof since they approximate sufficiently well the distances, at least in this situation.

Proof. — Let $\eta \geqslant 0$. Let $\theta = \theta(\eta) \geqslant 1$. Its exact value will be precised below. Let (H, Y) be an η-quasi-convex subgroup of G. Let $y \in Y$. We put $U = \mathrm{Stab}_G(y, 4\eta + 3) \cap H$. Note that since the action of G on X is proper, the subset U is finite. We claim that U is a generating set of H and that for every $h \in H$,

$$\frac{1}{\theta} d_U(1_G, h) - \theta \leqslant |y - hy| \leqslant \theta d_U(1_G, h).$$

Let $h \in H$. Let $\gamma\colon [0, L] \to X$ be a path of $\mathscr{P}$ joining y to hy. Let $m = \lfloor L \rfloor + 1$. We fix a partition $0 = t_0 \leqslant t_1 \leqslant \cdots \leqslant t_m = L$ of $[0, L]$ such that $|t_{m-1} - t_m| \leqslant 1$ and such that if $m \geqslant 2$, then for every $i \in \{0, \cdots, m-2\}$, we have $|t_i - t_{i+1}| = 1$. Let $i \in \{0, \cdots, m\}$. Denote $x_i = \gamma(t_i)$. Since (H, Y) is η-quasi-convex, there exist $h_i \in H$ such that $|h_i y - x_i| \leqslant 2\eta + 1$. Without loss of generality, we can take $h_0 = 1_G$ and $h_m = h$. By the triangle inequality,

$$|h_{i+1}y - h_i y| \leqslant |h_{i+1}y - x_{i+1}| + |x_{i+1} - x_i| + |x_i - h_i y|.$$

Note that $|x_i - x_{i+1}| \leqslant 1$. Consequently, $|h_i^{-1}h_{i+1}y - y| \leqslant 4\eta + 3$. Therefore $h_i^{-1}h_{i+1}$ belongs to U. We obtain

$$h = h_0^{-1}h_m = (h_0^{-1}h_1) \cdots (h_{m-1}^{-1}h_m).$$

Thus h can be written as a product of elements of U. Hence, the set U generates H. Besides, we have that $d_U(1_G, h) \leqslant m$. By construction of the partition, $m \leqslant L + 1$ and since γ is a (μ, ν)-quasi-geodesic, $L \leqslant \mu d(y, hy) + \nu$. Consequently,

$$|y - hy| \geqslant \frac{1}{\mu} d_U(1_G, h) - \frac{\nu}{\mu} - \frac{1}{\mu}.$$

Finally, let $h \in H$ and denote $m = d_U(1_G, h)$. By definition, there exist $u_1, \cdots, u_m \in U$ such that $h = u_1 \cdots u_m$. Applying the triangle inequality and the definition of U, we obtain

$$|y - hy| \leqslant \sum_{i=1}^{m} |y - u_i y| \leqslant (4\eta + 3) d_U(1_G, h).$$

Finally, we put $\theta = \max\left\{\mu, \frac{\nu}{\mu} + \frac{1}{\mu}, 4\eta + 3\right\}$. □

Lemma 1.5.6. — *Let $\eta \geqslant 0$. Let $H \leqslant G$ be an abelian subgroup. Let $Y \subset X$ be an H-invariant subset so that the action of H on Y is η-cobounded. Then, for every $h \in H$ and for every $y, z \in Y$,*

$$\Big||y - hy| - |z - hz|\Big| \leqslant 2\eta.$$

Proof. — Let $h \in H$. Let $y, z \in Y$. Since the action of H on Y is η-cobounded, there exists $k \in H$ such that $|z - ky| \leqslant \eta$. By the triangle inequality,

$$|y - hy| \leqslant |ky - khy| \leqslant |ky - z| + |z - hz| + |hz - khy|.$$

Since the subgroup H is abelian, $|hz - khy| = |z - ky|$. Thus, $|y - hy| \leqslant |z - hz| + 2\eta$. Finally, exchanging the roles of y and z, we obtain $|y - hy| \geqslant |z - hz| - 2\eta$. □

Next, we are going to check that we can obtain uniform quasi-isometric embeddings of $\mathbf{Z}$ in X via the orbit maps of quasi-convex elements of G that share the same constant. For this reason, we introduce the following definition:

Definition 1.5.7. — Let $g \in G$. Let $x \in X$. The *stable translation length* of g is

$$\|g\|^{\infty} = \limsup_{m \to \infty} \frac{1}{m} |g^m x - x|.$$

Note that $\|g\|^{\infty}$ does not depend on the choice of the point $x \in X$.

Remark 1.5.8. — Let $g \in G$. By subadditivity, for every $x \in X$, we have

$$\|g\|^{\infty} = \inf_{m \geqslant 1} \frac{1}{m} |g^m x - x| = \lim_{m \to \infty} \frac{1}{m} |g^m x - x|.$$

Lemma 1.5.9. — *Let $\eta \geqslant 0$. Let $g \in G$. Let $A \subset X$ be a $\langle g \rangle$-invariant subset so that the action of $\langle g \rangle$ on A is η-cobounded. The following statements are equivalent:*

(i) There exists $x \in X$ such that the orbit map $\mathbf{Z} \to X$, $m \mapsto g^m x$ is a quasi-isometric embedding.

(ii) $\|g\|^{\infty} > 0$.

(iii) There exists $\theta = \theta(\eta, g, A) \geqslant 1$ such that for every $a \in A$, the orbit map $\mathbf{Z} \to X$, $m \mapsto g^m a$ is a $(\theta, 0)$-quasi-isometric embedding.

Proof. — The implication $(iii) \Rightarrow (i)$ already holds.

$(i) \Rightarrow (ii)$. Assume that there exists $x \in X$ such that the orbit map $\mathbf{Z} \to X$, $m \mapsto g^m x$ is a quasi-isometric embedding. Then there exist $\kappa \geqslant 1$, $l \geqslant 0$ such that for every $m \geqslant 1$,

$$\frac{1}{\kappa} - \frac{l}{m} \leqslant \frac{1}{m}|x - g^m x| \leqslant \kappa + \frac{l}{m}.$$

Therefore, $\|g\|^\infty \geqslant \frac{1}{\kappa} > 0$.

$(ii) \Rightarrow (iii)$. Assume that $\|g\|^\infty > 0$. Let $\|g\|_A = \inf_{a \in A} |a - ga|$. Then we can define $\theta = \max\left\{\|g\|_A + 2\eta, \frac{1}{\|g\|^\infty}, 1\right\}$. Let $a \in A$. Applying the triangle inequality we obtain that for every $m \in \mathbf{Z}$, $|a - g^m a| \leqslant |a - ga||m|$. It follows from Lemma 1.5.6 that $|a - ga| \leqslant \|g\|_A + 2\eta$. Since $\|g\|^\infty = \inf_{n \in \mathbf{Z} - \{0\}} \frac{1}{|n|}|a - g^{|n|}a|$, we obtain that for every $m \in \mathbf{Z}$, $|a - g^m a| \geqslant \|g\|^\infty |m|$. Hence the orbit map $\mathbf{Z} \to X$, $m \mapsto g^m a$ is a $(\theta, 0)$-quasi-isometric embedding. □

Lemma 1.5.10. — *Let $\eta \geqslant 0$. Let (g, A) be an η-quasi-convex element of G. There exists $\theta = \theta(\eta, g, A) \geqslant 1$ such that for every $a \in A$, the orbit map $\mathbf{Z} \to X$, $m \mapsto g^m a$ is a $(\theta, 0)$-quasi-isometric embedding. Moreover, $\|g\|^\infty > 0$.*

Proof. — We are going to apply Lemma 1.5.5 and Lemma 1.5.9. Let $a \in A$. According to Lemma 1.5.5, there exist a finite generating set U of $\langle g \rangle$ such that the orbit map $\phi\colon (\langle g \rangle, d_U) \to X$, $h \mapsto ha$ is a quasi-isometric embedding. Furthermore, since g has infinite order, the map $\chi\colon \mathbf{Z} \to \langle g \rangle$, $m \mapsto g^m$ is an isomorphism. Let $V = \chi^{-1}(U)$. In particular $\chi\colon (\mathbf{Z}, d_V) \to (\langle g \rangle, d_U)$ is an isometry. Morover, the map $\psi\colon \mathbf{Z} \to (\mathbf{Z}, d_V)$ is a quasi-isometric embedding. Hence, the composition $\phi \circ \chi \circ \psi$ is a quasi-isometric embedding. Now both of the statements of the lemma follow from Lemma 1.5.9. □

We continue by upper bounding the length of a quasi-geodesic of X by the number of points of an orbit of a subgroup H of G that fall inside a precise neighbourhood of this quasi-geodesic, whenever the quasi-geodesic falls also inside a neighbourhood of that orbit.

Lemma 1.5.11. — *For every $\eta \geqslant 0$, $\kappa \geqslant 1$, $l \geqslant 0$, there exists $\theta \geqslant 1$ with the following property. Let $H \leqslant G$. Let $Y \subset X$ be an H-invariant subset such that the action of H on Y is η-cobounded. Let $y \in Y$. Let γ be a (κ, l)-quasi-geodesic of X such that $\gamma \subset Y^{+\eta}$. Let $U = \{u \in H\colon\ uy \in \gamma^{+2\eta+1}\}$. Then*

$$\ell(\gamma) \leqslant \theta|U|.$$

Proof. — Let $\eta \geqslant 0$, $\kappa \geqslant 1$, $l \geqslant 0$. Let $\theta = \theta(\eta, \kappa, l) \geqslant 1$. Its exact value will be precised below. Let H, Y, y, $\gamma\colon [0, L] \to X$ and U as in the statement. Let $m = \left\lfloor \frac{L}{\theta} \right\rfloor + 1$. We fix a partition $0 = t_0 \leqslant t_1 \leqslant \cdots \leqslant t_m = L$ of $[0, L]$ such that $|t_{m-1} - t_m| \leqslant \theta$ and such that if $m \geqslant 2$, then for every $i \in [\![0, m-2]\!]$, we have $|t_i - t_{i+1}| = \theta$. Hence $\ell(\gamma) = L \leqslant \theta m$. We prove that $m \leqslant |U|$. Let $i \in [\![0, m-1]\!]$. Denote $x_i = \gamma(t_i)$. Since the action of H on Y is η-cobounded and $\gamma \subset Y^{+\eta}$, for every $i \in [\![0, m-1]\!]$, there exists $h_i \in H$ such that $|x_i - h_i y| \leqslant 2\eta + 1$. In particular, $h_i \in U$. From now on we may assume that $m \geqslant 2$, otherwise there is nothing to show. Let $i, j \in [\![0, m-1]\!]$ such that $i \neq j$. We claim that $h_i \neq h_j$. The claim will follow when we show that $|h_i y - h_j y| > 0$. By the triangle inequality,

$$|h_i y - h_j y| \geqslant |x_i - x_j| - |x_i - h_i y| - |x_j - h_j y|.$$

Since γ is a (κ, l)-quasi-geodesic,

$$|x_i - x_j| \geqslant \frac{1}{\kappa}|t_i - t_j| - \frac{l}{\kappa}.$$

Since $i, j \in [\![0, m-1]\!]$, we have that $|t_i - t_j| \geqslant \theta$. To sum up,

$$|h_i y - h_j y| \geqslant \frac{\theta}{\kappa} - \frac{l}{\kappa} - 4\eta - 2.$$

Finally, we put $\theta = \kappa\left(\frac{l}{\kappa} + 4\eta + 2\right) + 1$. Hence, $|h_i y - h_j y| > 0$. In particular, we obtain $m \leqslant |U|$. □

The following fact is a direct consequence of the triangle inequality:

Lemma 1.5.12. — *Let $\eta \geqslant 0$. Let $H \leqslant G$. Let $Y \subset X$ be an H-invariant subset so that the action of H on Y is η-cobounded. Then, for every $y, z \in Y$, there exists $h \in H$ such that for every $r > 0$,*

$$h^{-1}\operatorname{Stab}_G(y, r)h \subset \operatorname{Stab}_G(z, r + 2\eta).$$

Finally, we show that there is a uniform threshold that ensures the existence of a uniformly short element in the intersection of any pair of quasi-convex subgroups of G that share the same constant.

Lemma 1.5.13. — *For every $\eta \geqslant 0$, there exists $\theta \geqslant 1$ with the following property. Let (H, Y) and (K, Z) be η-quasi-convex subgroups of G. If $\operatorname{diam}(Y \cap Z) > \theta$, then there exist $y \in Y \cap Z$ and $h \in H \cap K \cap \operatorname{Stab}_G(y, \theta) - \{1_G\}$.*

Proof. — Let $\eta \geqslant 0$. Let $\theta_0 = \theta_0(\eta, \mu, \nu) \geqslant 1$ be the constant of Lemma 1.5.11. Let $o \in Y$. We denote $W = \mathrm{Stab}_G(o, 6\eta + 2)$. Let $\theta_1 = \theta_0|W| + \theta_0$. Note that the constant θ_1 is finite since the action of G on X is proper. We put $\theta = 2\theta_1 + 4\eta + 2$. Let (H, Y) and (K, Z) be η-quasi-convex subgroups of G. Assume that $\mathrm{diam}(Y \cap Z) > \theta$. Since $\mathrm{diam}(Y \cap Z) > \theta_1$, there exist $y, z \in Y \cap Z$ such that $|y - z| > \theta_1$. Let $\beta \in \mathscr{P}$ joining y to z. Since $\ell(\beta) > \theta_1$, there exist $z' \in \beta$ and a subpath γ of β joining y to z' such that $\ell(\gamma) = \theta_1$. We denote $U = \{u \in H\colon\ uy \in \gamma^{+2\eta+1}\}$ and $V = \mathrm{Stab}_G(y, 4\eta + 2)$.

The first step is to construct a map $\phi\colon U \to V$. Let $u \in U$. By definition of U, there exists $x \in \gamma$ such that $|uy - x| \leqslant 2\eta + 1$. Since the subgroup (K, Z) is η-quasi-convex, there exists $k_u \in K$ such that $|x - k_u y| \leqslant 2\eta + 1$. By the triangle inequality,

$$|uy - k_u y| \leqslant |uy - x| + |x - k_u y|.$$

Consequently, $|u^{-1}k_u y - y| \leqslant 4\eta + 2$. Hence, $u^{-1}k_u \in V$. We define $\phi\colon U \to V$ to be the map that sends every $u \in U$ to $u^{-1}k_u \in V$.

Next, we show that the map $\phi\colon U \to V$ is not injective. Since Y is η-quasi-convex, we have that $\gamma \subset \beta \subset Y^{+\eta}$. It follows from Lemma 1.5.11 that $|U| \geqslant \frac{1}{\theta_0}\ell(\gamma)$. By hypothesis, $\ell(\gamma) = \theta_0|W| + \theta_0$. Since the action of H on Y is η-cobounded, it follows from Lemma 1.5.12 that there exists $h \in H$ such that $h^{-1}Vh \subset W$ and hence $|W| \geqslant |h^{-1}Vh| = |V|$. Consequently, $|U| > |V|$. Therefore, the map $\phi\colon U \to V$ is not injective.

Now we claim that $U \subset \mathrm{Stab}_G(y, \theta_1 + 2\eta + 1)$. Let $u \in U$. By definition of U, there exists $x \in \gamma$ such that $d|x - uy| \leqslant 2\eta + 1$. By the triangle inequality,

$$|y - uy| \leqslant |y - x| + |x - uy|.$$

Moreover, $|y - x| \leqslant \ell(\gamma) = \theta_1$. Hence $|y - uy| \leqslant \theta_1 + 2\eta + 1$.

Finally, since the map $\phi\colon U \to V$ is not injective, there exist $u_1, u_2 \in U$ such that $u_1 \neq u_2$ and $u_1^{-1}k_{u_1} = u_2^{-1}k_{u_2}$. In particular, $u_2u_1^{-1} \in H \cap K - \{1_G\}$. Further, according to the triangle inequality,

$$|y - u_2u_1^{-1}y| \leqslant |y - u_2y| + |u_2y - u_2u_1^{-1}y|.$$

It follows from the claim above that $|y - u_2u_1^{-1}y| \leqslant \theta$. Therefore, $u_2u_1^{-1} \in H \cap K \cap \mathrm{Stab}_G(y, \theta) - \{1_G\}$. □

We are ready to prove the proposition:

Proof of Proposition 1.5.2. — Let $\eta \geqslant 0$. Assume that G contains an η-quasi-convex element (g, A). We are going to determine the value of $\theta = \theta(\eta, g, A) \geqslant 1$. By Lemma 1.5.10, there exists $\theta_0 = \theta_0(\eta, g, A) \geqslant 1$ such that for every $a \in A$, the orbit map $\mathbf{Z} \to X$, $m \mapsto g^m a$ is a $(\theta_0, 0)$-quasi-isometric embedding. Let $\theta_1 = \theta_1(\eta) \geqslant 1$ be the constant of Lemma 1.5.13. Let $\theta_2 = \eta + \theta_0^2\theta_1$. Let $o \in A$. We denote $U = \mathrm{Stab}_G(o, 2\theta_2 + \eta + 1)$. Let $\theta = \max\{\theta_2, |U|\}$. Note that the constant θ is finite since the action of G on X is proper. Let (H, Y) be an η-quasi-convex subgroup of G.

(i) Let $u \in G$. Assume that $\mathrm{diam}(uA \cap Y) > \theta$. Let $a \in A$. We prove that $ua \in Y^{+\theta_2}$. Since $\mathscr{P}$ is G-invariant, the element (ugu^{-1}, uA) is η-quasi-convex. Since $\mathrm{diam}(uA \cap Y) > \theta_1$, according to Lemma 1.5.13, there exist $b \in A$ and $M \in \mathbf{Z} - \{0\}$ such that $ub \in uA \cap Y$ and $ug^M u^{-1} \in H \cap \mathrm{Stab}_G(ub, \theta_1)$. Since the action of $\langle g \rangle$ on A is η-cobounded, there exists $m \in \mathbf{Z}$ such that $|a - g^m b| \leqslant \eta$. By Euclid's division Lemma, there exist $q, r \in \mathbf{Z}$ such that $m = qM + r$ and $0 \leqslant r \leqslant |M| - 1$. By the triangle inequality,

$$d(ua, Y) \leqslant |ua - ug^{qM} b| \leqslant |ua - ug^m b| + |ug^m b - ug^{qM} b|.$$

Note that $|ua - ug^m b| = |a - g^m b| \leqslant \eta$. Moreover, it follows from Lemma 1.5.10 that

$$|ug^m b - ug^{qM} b| = |g^r b - b| \leqslant \theta_0 |r|.$$

Note also that $|r| \leqslant |M|$. Applying again Lemma 1.5.10, we obtain that $|M| \leqslant \theta_0 |g^M b - b|$. By Lemma 1.5.13, $|g^M b - b| = |ug^M u^{-1} ub - ub| \leqslant \theta_1$. Hence,

$$d(ua, Y) \leqslant \theta_2 \leqslant \theta.$$

(ii) Let $H \leqslant K \leqslant G$. We argue by contraposition. Assume that for every $k \in K$, we have $\mathrm{diam}(kA \cap Y) > \theta$. We prove that $[K : H] \leqslant |U|$. It follows from (i) that $KA \subset Y^{+\theta_2}$. Then there exists $y \in Y$ such that $|o - y| \leqslant \theta_2 + 1$. Since the action of H on Y is η-cobounded, we have that $Y \subset (Hy)^{+\eta}$. Hence $Ko \subset (Hy)^{+\theta_2+\eta}$. In particular, for every $k \in K$, there exists $h_k \in H$ such that $|ko - h_k y| \leqslant \theta_2 + \eta$. Let K' be a set of representatives of the set $H \backslash K$ of right cosets of H. Then the set $K'' = \{h_k^{-1} k \colon k \in K'\}$ is a set of representatives of $H \backslash K$. We claim that $K'' \subset U$.

Let $k \in K'$. By the triangle inequality,

$$|h_k^{-1}ko - o| = |ko - h_k o| \leqslant |ko - h_k y| + |h_k y - h_k o|.$$

Thus, $|h_k^{-1}ko - o| \leqslant 2\theta_2 + \eta + 1$. This proves the claim. Consequently,

$$[K : H] \leqslant |K''| \leqslant |U| \leqslant \theta.$$

□

1.6 Constricting elements

Hypothesis and conventions for this section. We fix:

- Constants $\mu \geqslant 1$ and ν, $\delta \geqslant 0$.
- A (μ, ν)-path system group $(G, X, \mathscr{P})$.
- A δ-constricting element (g, A).
- A δ-constricting map $\pi_A \colon X \to A$.

1.6.1 A G-invariant family

The set of G-translates of A is a G-invariant family of δ-constricting subsets. Indeed, consider the stabilizer $\operatorname{Stab}(A)$ of A and fix a set R_g of representatives of $G/\operatorname{Stab}(A)$. Let $u \in G$ and $u_0 \in R_g$ such that $uA = u_0 A$. The map $\pi_{uA} \colon X \to uA$ defined as

$$\forall\, x \in X, \quad \pi_{uA}(x) = u_0 \pi_A(u_0^{-1} x).$$

is then δ-constricting since $\mathscr{P}$ is G-invariant. Moreover, the element (ugu^{-1}, uA) is δ-constricting. To cope with the possible lack of $\langle ugu^{-1}\rangle$-equivariance of the map $\pi_{uA} \colon X \to uA$, we make the following observation:

Proposition 1.6.1. — *There exists $\theta \geqslant 0$ satisfying the following. Let $u \in G$. Then:*

(i) For every $x \in X$, we have $|\pi_{uA}(x) - u\pi_A(u^{-1}x)| \leqslant \delta$.

(ii) For every $Y \subset X$, we have $|\operatorname{diam}_{uA}(Y) - \operatorname{diam}(u\pi_A(u^{-1}Y))| \leqslant \theta$.

Proof. — Let $\theta_0 = \theta_0(\delta) \geqslant 0$ be the constant of Proposition 1.1.5. We put $\theta = 2\theta_0$. Let $u \in G$.

(i) Let $x \in X$. Denote $y = u^{-1}x$. Let $u_0 \in R_g$ such that $uA = u_0A$. We see that,

$$|\pi_{uA}(x) - u\pi_A(u^{-1}x)| = |u_0\pi_A(u_0^{-1}x) - u\pi_A(u^{-1}x)| = |\pi_A(u_0^{-1}uy) - u_0^{-1}u\pi_A(y)|.$$

Since $u_0^{-1}u \in Stab(A)$, it follows from Proposition 1.1.5 (2) *Coarse equivariance* that $|\pi_{uA}(x) - u\pi_A(u^{-1}x)| \leqslant \theta_0$.

(ii) Let $Y \subset X$. Let $y, y' \in Y$. By the triangle inequality,

$$\Big||\pi_{uA}(y) - \pi_{uA}(y')| - |u\pi_A(u^{-1}y) - u\pi_A(u^{-1}y')|\Big| \leqslant$$
$$|\pi_{uA}(y) - u\pi_A(u^{-1}y)| + |u\pi_A(u^{-1}y') - \pi_{uA}(y')|.$$

It follows from (i) that

$$\max\left\{|u\pi_{uA}(y) - u\pi_A(u^{-1}y)|, |u\pi_A(u^{-1}y') - \pi_{uA}(y')|\right\} \leqslant \theta_0.$$

Hence, we have $|\operatorname{diam}_{uA}(Y) - \operatorname{diam}(u\pi_A(u^{-1}Y))| \leqslant 2\theta_0$.

□

1.6.2 Finding a constricting element

The goal of this subsection is to combine Proposition 1.5.2 and Proposition 1.4.1. We suggest to compare (ii) below with the property (BS2) of the buffering sequences.

Proposition 1.6.2. — *Let $\eta \geqslant 0$. There exists $\theta \geqslant 1$ satisfying the following. Let (H, Y) be an η-quasi-convex subgroup of G. Then:*

(i) For every $u \in G$, if $\operatorname{diam}_{uA}(Y) > \theta$, then $uA \subset Y^{+\theta}$.

(ii) Let $H \leqslant K \leqslant G$. If $[K : H] > \theta$, then there exists $k \in K$ such that $\operatorname{diam}_{kA}(Y) \leqslant \theta$.

Proof. — Let $\eta \geqslant 0$. Let $\theta = \theta(\eta) \geqslant 1$. Its exact value will be precised below. It follows from Proposition 1.1.5 (6) *Morseness* and (7) *Coarse invariance* that there exists $\theta_0 \geqslant 0$ such that the element (g, A) is θ_0-quasi-convex. Let $\theta_1 = \max\{\eta, \theta_0\}$. By Proposition 1.4.1, there exist $\theta_2 \geqslant 0$, $\zeta \geqslant 0$ depending both on θ_1 such that for every $u \in G$ and for every θ_1-quasi-convex subset $Y \subset X$, we have

$$\operatorname{diam}_{uA}(Y) - \zeta \leqslant \operatorname{diam}(uA^{+\theta_2} \cap Y) \leqslant \operatorname{diam}_{uA}(Y) + \zeta.$$

According to Proposition 1.1.5 (6) *Morseness* and (7) *Coarse invariance*, there exist $\theta_3 = \theta_3(\theta_2) \geqslant 0$ such that the element $(g, A^{+\theta_2})$ is θ_3-quasi-convex. Let $\theta_4 = \max\{\eta, \theta_3\}$. Let $\theta_5 = \theta_5(\theta_4, g, A) \geqslant 1$ be the constant of Proposition 1.5.2. Finally, we put $\theta = \theta_5 + \zeta$. Let (H, Y) be an η-quasi-convex subgroup of G.

(i) Let $u \in G$. Assume that $\mathrm{diam}_{uA}(Y) > \theta$. According to Proposition 1.4.1, we have $\mathrm{diam}(uA^{+\theta_2} \cap Y) > \theta_5$ and according to Proposition 1.5.2 (i) this implies that $uA \subset Y^{+\theta_5} \subset Y^{+\theta}$.

(ii) Let $H \leqslant K \leqslant G$. We argue by contraposition. Assume that for every $k \in K$, we have $\mathrm{diam}_{kA}(Y) > \theta$. According to Proposition 1.4.1, for every $k \in K$, we have $\mathrm{diam}(kA^{+\theta_2} \cap Y) > \theta_5$ and according to Proposition 1.5.2 (ii) this implies that $[K : H] \leqslant \theta_5 \leqslant \theta$.

□

1.6.3 Elementary closures

The elementary closure of (g, A) could be thought as the set of elements $u \in G$ such that uA is "parallel" to A:

Definition 1.6.3. — The *elementary closure of* (g, A) *in* G is defined as

$$E(g, A) = \{u \in G\colon\ d_{\mathrm{Haus}}(uA, A) < \infty\}.$$

Observe that $E(g, A)$ is a subgroup of G since d_{Haus} is a pseudo-distance.

This subsection is devoted to provide a further description $E(g, A)$. We suggest to compare the proposition below with the property (BS1) of the buffering sequences.

Proposition 1.6.4. — *There exists $\theta \geqslant 1$ satisfying the following:*

(i) For every $u \in G$, we have

$$\max\{\mathrm{diam}_{uA}(A), \mathrm{diam}_A(uA)\} > \theta \quad \Longleftrightarrow \quad d_{\mathrm{Haus}}(uA, A) \leqslant \theta.$$

(ii) $E(g, A) = \{u \in G\colon\ d_{\mathrm{Haus}}(uA, A) \leqslant \theta\}$.

(iii) $[E(g, A) : \langle g\rangle] \leqslant \theta$.

Proof. — Let $\theta_0 \geqslant 0$ be the constant of Proposition 1.6.1. According to Proposition 1.1.5 (6) *Morseness*, there exists $\theta_1 \geqslant 0$ such that the element (g, A) is θ_1-quasi-convex. Let $\theta_2 = \theta_2(\theta_1) \geqslant 1$ be the constant of Proposition 1.6.2. We put $\theta = \theta_0 + \theta_2$.

Claim 1.6.5. — Let $u \in G$. If $d_{\text{Haus}}(uA, A) < \infty$, then $\text{diam}_{uA}(A) = \infty$.

Let $u \in G$. Assumme that $d_{\text{Haus}}(uA, A) < \infty$ and denote $\varepsilon = d_{\text{Haus}}(uA, A) + 1$. By Proposition 1.4.1, there exist θ_3, $\zeta \geqslant 0$ such that for every $u \in G$ we have

$$\text{diam}_{uA}(A) - \zeta \leqslant \text{diam}(uA^{+\theta_3} \cap A^{+\varepsilon}) \leqslant \text{diam}_{uA}(A) + \zeta.$$

Note that $uA \subset uA^{+\theta_3} \cap A^{+\varepsilon}$ and $\text{diam}(uA) = \text{diam}(A)$. Since the action of G on X is proper and since the element g has infinite order, we have that $\text{diam}(A) = \infty$. Consequently, we have $\text{diam}(uA^{+\theta_3} \cap A^{+\varepsilon}) = \infty$. Finally, it follows from Proposition 1.4.1 that $\text{diam}_{uA}(A) = \infty$. This proves the claim.

(i) Let $u \in G$. Assume that $\max\{\text{diam}_{uA}(A), \text{diam}_A(uA)\} > \theta$. By Proposition 1.6.1,

$$\text{diam}_{u^{-1}A}(A) \geqslant \text{diam}_A(u^{-1}\pi_A(uA)) - \theta_0.$$

Hence, $\text{diam}_{u^{-1}A}(A) > \theta_2$. It follows from Proposition 1.6.2 (i) that $uA \subset A^{+\theta}$ and $u^{-1}A \subset A^{+\theta}$. Hence $d_{\text{Haus}}(uA, A) \leqslant \theta$. The converse follows from the claim above.

(ii) This follows from (i) and the claim above.

(iii) This follows from (i), (ii) and Proposition 1.6.2 (ii).

□

Finally, we obtain an algebraic description of $E(g, A)$.

Corollary 1.6.6. — *There exist $\theta \geqslant 1$ and $M \in [\![1, \theta]\!]$ such that for every $u \in G$, the following statements are equivalent:*

(i) $u \in E(g, A)$.

(ii) There exists $p \in \{-1, 1\}$ such that $ug^M u^{-1} = g^{pM}$.

(iii) There exist $m, n \in \mathbf{Z} - \{0\}$ such that $ug^m u^{-1} = g^n$.

Further, let $E^+(g, A) = \{u \in G \colon ug^M u^{-1} = g^M\}$. Then $[E(g, A) : E^+(g, A)] \leqslant 2$.

Proof. — By Proposition 1.6.4 *(ii)*, there exists $\theta_0 \geqslant 1$ such that $[E(g, A) : \langle g \rangle] \leqslant \theta_0$. Let $\theta = \theta_0!$ We construct $M \in [\![1, \theta]\!]$. First, we claim that there exists a subgroup

$K \leqslant \langle g\rangle$ such that $K \trianglelefteq E(g,A)$ and $[E(g,A):K] \leqslant \theta$. Consider the natural action of $E(g,A)$ by right multiplication on the set $\langle g\rangle\backslash E(g,A)$ of right cosets of $\langle g\rangle$. This gives an homomorphism $\phi\colon E(g,A) \to \mathrm{Sym}(\langle g\rangle\backslash E(g,A))$. Choose $K = Ker(\phi)$. Note that $\langle g\rangle = \{h \in E(g,A)\colon \phi(h)(\langle g\rangle)\} = \langle g\rangle$. Thus, $K \leqslant \langle g\rangle$. Morover, $K \trianglelefteq E(g,A)$. Further, we have that $|Sym(\langle g\rangle\backslash E(g,A))| = [E(g,A):\langle g\rangle]!$ and hence $[E(g,A):K]$ divides $[E(g,A):\langle g\rangle]!$ Therefore, $[E(g,A):K] \leqslant \theta$. This proves the claim. Now, since the element g has infinite order, the subgroup $E(g,A)$ is infinite. Hence, since $[E(g,A):K] < \infty$ there exists $M \geqslant 1$ such that $K = \langle g^M\rangle$. Finally, we remark that M is equal to the order of the element $\phi(g)$. Hence, $M \leqslant \theta$.

Let $u \in G$. The implication $(ii) \Rightarrow (iii)$ already holds.

$(i) \Rightarrow (ii)$. Assume that $u \in E(g,A)$. Since the subgroup $\langle g^M\rangle$ is normal in $E(g,A)$, there exists $p \in \mathbf{Z}$ such that $ug^Mu^{-1} = g^{pM}$. In particular,

$$\langle g^M\rangle = u\langle g^M\rangle u^{-1} = \langle ug^Mu^{-1}\rangle = \langle g^{pM}\rangle.$$

Hence, if $p \notin \{-1,+1\}$, then $\langle g^M\rangle \not\subset \langle g^{pM}\rangle$. Contradiction.

$(iii) \Rightarrow (i)$. Assume that there exist $m,n \in \mathbf{Z} - \{0\}$ such that $ug^mu^{-1} = g^n$. Since both $\langle g^m\rangle$ and $\langle g^n\rangle$ have finite index in $\langle g\rangle$, there exist $\zeta \geqslant 0$ the actions of $\langle ug^mu^{-1}\rangle$ on uA and of $\langle g^n\rangle$ on A are both ζ-cobounded. Let $x \in uA$ and $y \in A$. We obtain $d_{\mathrm{Haus}}(uA,A) \leqslant \zeta + |x-y|$. Hence $d_{\mathrm{Haus}}(uA,A) < \infty$.

Finally, let $E^+(g,A) = \{u \in G\colon\ ug^Mu^{-1} = g^M\}$. We prove that $[E(g,A):E^+(g,A)] \leqslant 2$. It is enough to assume that $E(g,A) \neq E^+(g,A)$. Let $u,v \in E(g,A) - E^+(g,A)$. We show that $v^{-1}u \in E^+(g,A)$. Since $ug^Mu^{-1} = vg^Mv^{-1} = g^{-M}$, we have $v^{-1}ug^Mu^{-1}v = v^{-1}g^{-M}v = g^M$ and therefore $v^{-1}u \in E^+(g,A)$. Hence $[E(g,A):E^+(g,A)] = 2$ ☐

1.6.4 Forcing a geometric separation

In this subsection, we build large powers of our constricting element (g,A) to produce a translate Y' of a subset Y so that the distance between their projections to a preferred G-translate of A is large. We will do it in two different ways. We will apply these results to verify (BS4) in the construction of buffering sequences. Our main tool will be:

Lemma 1.6.7. — *There exists $\theta \geqslant 0$ such that for every $x,x' \in X$ and for every $m \in \mathbf{Z}$,*

$$|x - g^m x'|_A \geqslant |m|\,\|g\|^\infty - |x-x'|_A - \theta.$$

Proof. — Let $\theta = \theta(\delta) \geqslant 0$ be the constant of Proposition 1.1.5. Let $x, x' \in X$. Let $m \in \mathbf{Z}$. If $m = 0$, then there is nothing to do. Assume that $m \neq 0$. By the triangle inequality,

$$|x - g^m x'|_A \geqslant |\pi_A(x) - g^m \pi_A(x)| - |x - x'|_A - |g^m \pi_A(x') - \pi_A(g^m x')|.$$

Note that

$$\frac{1}{|m|}|\pi_A(x) - g^m \pi_A(x)| \geqslant \inf_{n \geqslant 1} \frac{1}{n}|\pi_A(x) - g^n \pi_A(x)| = \|g\|^\infty .$$

By Proposition 1.1.5 (2) *Coarse equivariance*, we have $|g^m \pi_A(x') - \pi_A(g^m x')| \leqslant \theta$. Therefore, we have $|x - g^m x'|_A \geqslant |m| \, \|g\|^\infty - |x - x'|_A - \theta$. □

The first way of forcing a geometric separation will be applied to the study of the relative exponential growth rates:

Proposition 1.6.8. — *For every ε, $\theta \geqslant 0$, there exists $M \geqslant 1$ with the following property. Let $H \leqslant G$ be a subgroup. Let $Y \subset X$ be an H-invariant subset. If $\operatorname{diam}_A(Y) \leqslant \varepsilon$, then for every $u \in \langle g^M, H \cap E(g, A)\rangle - H \cap E(g, A)$, we have $d_A(Y, uY) > \theta$.*

Proof. — Let ε, $\theta \geqslant 0$. Let $\theta_0 \geqslant 0$ be the constant of Proposition 1.1.5. By Lemma 1.6.7, there exists $\theta_1 \geqslant 0$ such that for every $x, x' \in X$ and for every $m \in \mathbf{Z}$,

$$|x - g^m x'|_A \geqslant |m| \, \|g\|^\infty - |x - x'|_A - \theta_1.$$

Combining Lemma 1.5.10 and Proposition 1.1.5 (6) *Morseness*, we obtain $\|g\|^\infty > 0$. According to Corollary 1.6.6, there exists $M_0 \geqslant 1$ such that

$$E(g, A) = \left\{ u \in G \colon \exists p \in \{-1, +1\} \; u g^{M_0} u^{-1} = g^{pM_0} \right\}.$$

Let $m_0 > \frac{\theta - 2\varepsilon - 2\theta_0 - \theta_1}{M_0 \|g\|^\infty}$. We put $M = M_0 m_0$.

Let $H \leqslant G$ be a subgroup. Let $Y \subset X$ be an H-invariant subset. Assume that $\operatorname{diam}_A(Y) \leqslant \varepsilon$. Let $u \in \langle g^M, H \cap E(g, A)\rangle - H \cap E(g, A)$ and $y, y' \in Y$. It follows from Corollary 1.6.6 that there exists $n \in \mathbf{Z}$ multiple of M and $f \in H \cap E(g, A)$ such that $u = g^n f$. By the triangle inequality,

$$|y - g^n f y'|_A \geqslant |y - g^n y'|_A - |\pi_A(g^n y') - g^n \pi_A(y')| - |y' - f y'|_A - |g^n \pi_A(f y') - \pi_A(g^n f y')|.$$

By Lemma 1.6.7,

$$|y - g^n y'|_A \geqslant |n| \, \|g\|^\infty - |y - y'|_A - \theta_1$$

Note that $u \notin H \cap E(g, A)$ implies $n \neq 0$. Hence $|n| \geqslant |M|$. Since $f \in H$ and $\operatorname{diam}_A(Y) \leqslant \varepsilon$,

$$\max\{|y - y'|_A, |y' - fy'|_A\} \leqslant \varepsilon.$$

By Proposition 1.1.5 (2) *Coarse equivariance*,

$$\max\{|\pi_A(g^n y') - g^n \pi_A(y')|, |g^n \pi_A(fy') - \pi_A(g^n fy')|\} \leqslant \theta_0.$$

Since the elements y, y' are arbitrary, we obtain $d_A(Y, uY) > \theta$. □

The second way of forcing a geometric separation will be applied to the study of the quotient exponential growth rates:

Proposition 1.6.9. — *For every ε, $\theta \geqslant 0$, there exist $M \geqslant 1$ and $f\colon G \times X \to \{1_G, g^M\}$ with the following property. Let $Y \subset X$ be subset. If $\operatorname{diam}_A(Y) \leqslant \varepsilon$, then for every $u \in G$ and for every $y \in Y$, we have $d_{uA}(y, uf(u,y)Y) > \theta$.*

Proof. — Let ε, $\theta \geqslant 0$. Let $\theta_0 \geqslant 0$ be the constant of Proposition 1.6.1. By Lemma 1.6.7, there exists $\theta_1 \geqslant 0$ such that for every $x, x' \in X$ and for every $m \in \mathbf{Z}$,

$$|x - g^m x'|_A \geqslant |m| \, \|g\|^\infty - |x - x'|_A - \theta_1.$$

Combining Lemma 1.5.10 and Proposition 1.1.5 (6) *Morseness*, we obtain $\|g\|^\infty > 0$. We put $M > \frac{2\theta + 2\varepsilon + 8\theta_0 + \theta_1}{\|g\|^\infty}$. Then, for every $u \in G$ and for every $x \in X$, there exists $f(u,x) \in \{1_G, g^M\}$ such that $|u^{-1}x - f(u,x)|_A > \theta + \varepsilon + 4\theta_0$: if $|u^{-1}x - x|_A > \theta + \varepsilon + 4\theta_0$, we choose $f(u,x) = 1_G$, otherwise we choose $f(u,x) = g^M$. This defines $f\colon G \times X \to \{1_G, g^M\}$.

Let $Y \subset X$ be a subset. Assume that $\operatorname{diam}_A(Y) \leqslant \varepsilon$. Let $u \in G$. Let $y, y' \in Y$. By abuse of notation, we write f instead of $f(u,y)$. By the triangle inequality,

$$\begin{aligned}
|y - ufy'|_{uA} &\geqslant |y - ufy|_{uA} - |ufy - ufy'|_{uA}, \\
|y - ufy|_{uA} &\geqslant |u^{-1}y - fy|_A - |\pi_{uA}(y) - u\pi_A(u^{-1}y)| - |\pi_{uA}(ufy) - u\pi_A(fy)|, \\
|ufy - ufy'|_{uA} &\leqslant |\pi_{uA}(ufy) - uf\pi_A(y)| + |y - y'|_A + |uf\pi_A(y') - \pi_{uA}(ufy')|.
\end{aligned}$$

By hypothesis, $|u^{-1}y - fy|_A > \theta + \varepsilon + 4\theta_0$ and $|y - y'|_A \leqslant \operatorname{diam}_A(Y) \leqslant \varepsilon$. By Proposition 1.6.1,

$$\max\{|\pi_{uA}(y) - u\pi_A(u^{-1}y)|, |\pi_{uA}(ufy) - u\pi_A(fy)|\} \leqslant \theta_0.$$

$$\max\{|\pi_{uA}(ufy) - uf\pi_A(y)|, |uf\pi_A(y') - \pi_{uA}(ufy')|\} \leqslant \theta_0.$$

Since the element y' is arbitrary, we obtain $d_{uA}(y, ufY) > \theta$. □

1.7 Growth of quasi-convex subgroups

In this section, our first goal is to prove Theorem 0.5.8. This result can be deduced from Proposition 1.2.1 and Proposition 1.7.1 below. Our second goal is to prove Theorem 0.5.13. This result can be deduced from Proposition 1.2.4 and Proposition 1.7.3 below.

Hypothesis and conventions for this section. We fix:

- Constants $\mu \geqslant 1$ and ν, δ, $\eta \geqslant 0$.
- A (μ, ν)-path system group $(G, X, \mathscr{P})$.
- A δ-constricting element (g_0, A_0).
- An infinite index η-quasi-convex subgroup (H, Y) of G.

We are going to replace the axis A_0 for $A_0' = E(g_0, A_0)A_0$. As a consequence of Proposition 1.6.4 (ii), we have $d_{\text{Haus}}(A_0, A_0') < \infty$. Up to replacing δ for a larger constant, it follows from Proposition 1.1.5 (7) *Coarse invariance* and Corollary 1.6.6 that the element (g_0, A_0') is δ-constricting. By abuse of notation, we still denote $A_0 = A_0'$. In this new setting, for every $k \in E(g_0, A_0)$, we have $kA_0 = A_0$.

Let $\theta_0 = \theta_0(\delta, \eta) \geqslant 1$ be the constant of Proposition 1.6.2. Since $[G : H] = \infty$, it follows from Proposition 1.6.2 (ii) that there exist $u \in G$ such that $\text{diam}_{uA_0}(Y) \leqslant \theta_0$. We denote $(g, A) = (ug_0u^{-1}, uA_0)$.

Proposition 1.7.1 (Theorem 0.5.10)**. —** *There exist $M \geqslant 1$ such that the natural homomorphism $H *_{H\cap E(g,A)} \langle g^M, H \cap E(g, A)\rangle \to G$ is injective.*

Remark 1.7.2. — It follows from Proposition 1.4.1 and Proposition 1.6.4 that the subgroup $H \cap E(g, A)$ is finite. By Proposition 1.6.4, the subgroup $E(g, A)$ is a finite extension of $\langle g\rangle$. Hence the proposition proves Theorem 0.5.10. Since g has infinite order, the finite subgroup $H \cap E(g, A)$ is a proper subgroup of $\langle g^M, H \cap E(g, A)\rangle$. Hence we can apply Proposition 1.2.1 to deduce Theorem 0.5.8.

Proof. — Let $\theta_1 = \theta_1(\delta) \geqslant 0$ be the constant of Proposition 1.6.1. Let $\varepsilon = \max\{\theta_0 + 2\theta_1, d(A, Y)\}$. Let $L = L(\delta, \varepsilon, 0) \geqslant 0$ be the constant of Corollary 1.3.4. By Proposition 1.6.8, there exists $M \geqslant 1$ such that for every $u \in \langle g^M, H \cap E(g, A)\rangle - H \cap E(g, A)$, we have $d_A(Y, uY) > L - 2\theta_1$.

Let $\phi\colon H *_{H\cap E(g,A)} \langle g^M, H \cap E(g, A)\rangle \to G$ be the natural homomorphism. Let $w \in H *_{H\cap E(g,A)} \langle g^M, H \cap E(g, A)\rangle$ such that $w \neq 1$. We are going to prove that $\phi(w) \neq 1$. Note that the homomorphisms $\phi_{|H}$ and $\phi_{|\langle g^M, H\cap E(g,A)\rangle}$ are injective. If $w \in H \cup \langle g^M, H \cap E(g, A)\rangle$, then $\phi(w) \neq 1$. Assume that $w \notin H \cup \langle g^M, H \cap E(g, A)\rangle$. Note that if there exists a conjugate w' of w such that $\phi(w') \neq 1$, then $\phi(w) \neq 1$. Up to replacing w by a cyclic conjugate, there exist $n \geqslant 1$ and a sequence $h_1, k_1, \cdots, h_n, k_n \in G$ such that $w = h_1k_1 \cdots h_nk_n$ and such that for every $i \in \{1, \cdots, n\}$ we have $h_i \in H - H \cap E(g, A)$ and $k_i \in \langle g^M, H \cap E(g, A)\rangle - H \cap E(g, A)$. For every $i \in [\![1, n]\!]$, we denote $u_i = h_1k_1 \cdots h_i$ and $v_i = h_1k_1 \cdots h_ik_i$. We also denote $v_0 = 1_G$.

We are going to prove that the sequence $v_0Y, u_1A, v_1Y, \cdots, u_nA, v_nY$ is (δ, ε, L)-buffering on $\{u_iA\}$ and then apply Corollary 1.3.4. Let $i \in [\![1, n]\!]$. Let us prove (BS1). Assume for a moment that $i \neq n$. Since we had modified the axis A_0 above, for every $j \in [\![1, n]\!]$, we have $k_jA = A$. Hence

$$\pi_{u_iA}(u_{i+1}A) = \pi_{v_iA}(u_{i+1}A),$$
$$\pi_{u_{i+1}A}(u_iA) = \pi_{u_{i+1}A}(v_iA).$$

By Proposition 1.6.1,

$$\operatorname{diam}_{v_iA}(u_{i+1}A) \leqslant \operatorname{diam}(v_i\pi_A(h_iA)) + \theta_1,$$
$$\operatorname{diam}_{u_{i+1}A}(v_iA) \leqslant \operatorname{diam}(u_{i+1}\pi_A(h_i^{-1}A)) + \theta_1,$$
$$\operatorname{diam}_A(h_i^{-1}A) \leqslant \operatorname{diam}_{h_iA}(A) + \theta_1.$$

By Proposition 1.6.4 (i) and (ii), for every $u \notin E(g, A)$, we have $\max\{\operatorname{diam}_A(uA), \operatorname{diam}_{uA}(A)\} \leqslant \theta_0$. Consequently,

$$\max\{\operatorname{diam}_{u_iA}(u_{i+1}A), \operatorname{diam}_{u_{i+1}A}(u_iA)\} \leqslant \theta_0 + 2\theta_1 \leqslant \varepsilon.$$

Let us prove (BS2). Note that,

$$\pi_{u_iA}(v_{i-1}Y) = \pi_{u_iA}(u_iY),$$

$$\pi_{u_iA}(v_iY) = \pi_{v_iA}(v_iY).$$

By Proposition 1.6.1,

$$\operatorname{diam}_{u_iA}(u_iY) \leqslant \operatorname{diam}(u_i\pi_A(Y)) + \theta_1,$$
$$\operatorname{diam}_{v_iA}(v_iY) \leqslant \operatorname{diam}(v_i\pi_A(Y)) + \theta_1.$$

Since $\operatorname{diam}_A(Y) \leqslant \theta_0$, we obtain

$$\max\{\operatorname{diam}_{u_iA}(v_{i-1}Y), \operatorname{diam}_{u_iA}(v_iY)\} \leqslant \theta_0 + \theta_1 \leqslant \varepsilon.$$

Let us prove (BS3). We have,

$$\max\{d(u_iA, v_{i-1}Y), d(u_iA, v_iY)\} = \max\{d(u_iA, u_iY), d(v_iA, v_iY)\} \leqslant d(A, Y) \leqslant \varepsilon.$$

Let us prove (BS4). It follows from Proposition 1.6.1 (i) that,

$$d_{u_iA}(v_{i-1}Y, v_iY) \geqslant d_A(Y, k_iY) - 2\theta_1.$$

By the choice of M, we have $d_A(Y, k_iY) > L + 2\theta_1$. Hence, we have $d_{u_iA}(v_{i-1}Y, v_iY) \geqslant L$. This proves that the sequence $v_0Y, u_1A, v_1Y, \cdots, u_nA, v_nY$ is (δ, ε, L)-buffering on $\{u_iA\}$. It follows from Corollary 1.3.4 that $d_{u_nA}(Y, \phi(w)Y) > 0$. Hence, $\phi(w) \neq 1$.

□

Recall that given $\phi\colon G \to G$, we say that G is *ϕ-coarsely G/H* if there exist $\theta \geqslant 0$, $x \in X$ satisfying the following conditions:

(CQ1) For every $u, v \in G$, if $\phi(u)H = \phi(v)H$, then $|\phi(u)x - \phi(v)x| \leqslant \theta$.

(CQ2) For every $u \in G$, $|ux - \phi(u)x| \leqslant \theta$.

Proposition 1.7.3. — *There exist $M \geqslant 1$ and a map $f\colon G \to \{1_G, g^M\}$ with the following property. Let $\phi\colon G \to G$, $u \mapsto uf_u$. Then G is ϕ-coarsely G/H.*

We prove some preliminar lemmas.

Lemma 1.7.4. — *There exists $\theta \geqslant 0$ such that for every $m \in \mathbf{Z}$, we have* $\operatorname{diam}_A(g^mY) \leqslant \theta$.

Proof. — Let $\theta_1 \geqslant 0$ be the constant of Proposition 1.1.5. We put $\theta = \theta_0 + 2\theta_1$. Let $m \in \mathbf{Z}$.

Let $x, x' \in Y$. By the triangle inequality,

$$|g^m x - g^m x'|_A \leqslant |\pi_A(g^m x) - g^m \pi_A(x)| + |x - x'|_A + |g^m \pi_A(x') - \pi_A(g^m x')|.$$

By Proposition 1.1.5 (2) *Coarse equivariance*,

$$\max\{|\pi_A(g^m x) - g^m \pi_A(x)|, |g^m \pi_A(x') - \pi_A(g^m x')|\} \leqslant \theta_1.$$

Moreover, we have $|x - x'|_A \leqslant \operatorname{diam}_A(Y) \leqslant \theta_0$. Since x, x' are arbitrary, we obtain $\operatorname{diam}_A(g^m Y) \leqslant \theta_0 + 2\theta_1$. □

Lemma 1.7.5. — *For every $\varepsilon \geqslant 0$, there exists $\theta \geqslant 0$ with the following property. Let $A_1, A_2 \subset X$ be δ-constricting subsets such that $d_{\text{Haus}}(A_1, A_2) \leqslant \varepsilon$. Let $x \in A_1^{+\varepsilon}$ and $y \in A_2^{+\varepsilon}$ such that $|x - y|_{A_1} \leqslant \varepsilon$. Then $|x - y| \leqslant \theta$.*

Proof. — Let $\theta_1 \geqslant 0$ be the constant of Proposition 1.1.5. Let $\varepsilon \geqslant 0$. Let $\theta \geqslant 0$. Its exact value will be precised below. Let $A_1, A_2 \subset X$ be δ-constricting subsets such that $d_{\text{Haus}}(A_1, A_2) \leqslant \varepsilon$. Let $x \in A_1^{+\varepsilon}$ and $y \in A_2^{+\varepsilon}$ such that $|x - y|_{A_1} \leqslant \varepsilon$. By the triangle inequality,

$$|x - y| \leqslant |x - \pi_{A_1}(x)| + |x - y|_{A_1} + |\pi_{A_1}(y) - y|.$$

Since $x, y \in A_1^{+2\varepsilon+1}$, it follows from Proposition 1.1.5 (1) *Coarse nearest-point projection* that

$$\max\{|x - \pi_{A_1}(x)|, |\pi_{A_1}(y) - y|\} \leqslant \mu(2\varepsilon + 1) + \theta_1.$$

Finally, we put $\theta = \varepsilon + 2\mu(2\varepsilon + 1) + 2\theta_1$. □

We are ready to prove Proposition 1.7.3:

Proof of Proposition 1.7.3. — Let $\theta_1 \geqslant 0$ be the constant of Proposition 1.6.1. Let $\theta_2 \geqslant 0$ be the constant of Proposition 1.6.4. Let $\theta_3 \geqslant 0$ be the constant of Lemma 1.7.4. Let $\varepsilon = \max\{\theta_2 + 2\theta_1, \theta_1 + \theta_3, d(A, Y) + 1\}$. In particular, there exists $y \in A^{+\varepsilon} \cap Y$. Let $\theta_4 = \theta_4(\delta, \varepsilon) \geqslant 0$ be the constant of Proposition 1.3.2. By Proposition 1.6.9, there exist $M \geqslant 1$ and $f \colon G \to \{1_G, g^M\}$ such that for every $u \in G$, we have $d_{uA}(y, uf(u)Y) > \theta_4$. For every $u \in G$, we denote $f_u = f(u)$ and we put $\phi \colon G \to G, u \mapsto uf_u$. Let $\theta_5 = \theta_5(\varepsilon) \geqslant 0$ be the constant of Lemma 1.7.5. We put $\theta = \max\{|y - g^M y|, \theta_5\}$. We are going to prove that G is ϕ-coarsely G/H with respect to y and θ.

In order to prove (CQ1), we just need to observe that for every $u \in G$, we have

$$|uy - uf_u y| = |y - f_u y| \leqslant |y - g^M y| \leqslant \theta.$$

Let us prove (CQ2). Let $u, v \in G$. Assume that $uf_u H = vf_v H$. We claim that $d_{\text{Haus}}(uA, vA) \leqslant \theta_2$. By Proposition 1.6.4 (i), it suffices to prove that

$$\max\{\text{diam}_{v^{-1}uA}(A), \text{diam}_A(v^{-1}uA)\} > \theta_2.$$

We argue by contradiction. Assume instead that $\max\{\text{diam}_{v^{-1}uA}(A), \text{diam}_A(v^{-1}uA)\} \leqslant \theta_2$. We are going to prove that the sequence uA, uf_uY, vA is $(\delta, \varepsilon, 0)$-buffering on $\{uA, vA\}$ and then apply Proposition 1.3.2. Note that the condition (BS4) is void in this case. Let us prove (BS1). By Proposition 1.6.1,

$$\begin{aligned} \text{diam}_{uA}(vA) &\leqslant \text{diam}(u\pi_A(u^{-1}vA)) + \theta_1, \\ \text{diam}_{vA}(uA) &\leqslant \text{diam}(v\pi_A(v^{-1}uA)) + \theta_1, \\ \text{diam}_A(u^{-1}vA) &\leqslant \text{diam}_{v^{-1}uA}(A) + \theta_1. \end{aligned}$$

Hence,

$$\max\{\text{diam}_{uA}(vA), \text{diam}_{vA}(uA)\} \leqslant \theta_2 + 2\theta_1 \leqslant \varepsilon.$$

Let us prove (BS2). By Proposition 1.6.1,

$$\begin{aligned} \text{diam}_{uA}(uf_uY) &\leqslant \text{diam}(u\pi_A(f_uY)) + \theta_1, \\ \text{diam}_{vA}(vf_vY) &\leqslant \text{diam}(v\pi_A(f_vY)) + \theta_1. \end{aligned}$$

By Lemma 1.7.4, we have $\max\{\text{diam}_A(f_uY), \text{diam}_A(f_vY)\} \leqslant \theta_3$. Hence,

$$\max\{\text{diam}_{uA}(uf_uY), \text{diam}_{vA}(vf_vY)\} \leqslant \theta_1 + \theta_3 \leqslant \varepsilon.$$

Let us prove (BS3). The hypothesis $uf_uH = vf_vH$ implies $uf_uY = vf_vY$ and therefore

$$\max\{d(uA, uf_uY), d(vA, uf_uY)\} = \max\{d(uA, uf_uY), d(vA, vf_vY)\} = d(A, Y) \leqslant \varepsilon.$$

Hence, the sequence uA, uf_uY, vA is $(\delta, \varepsilon, 0)$-buffering on $\{uA, vA\}$. It follows from Proposition 1.3.2 that

$$\min\{d_{uA}(y, uf_uY), d_{vA}(y, uf_uY)\} \leqslant \theta_4.$$

However, by construction,

$$\min\{d_{uA}(y, uf_uY), d_{vA}(y, uf_uY)\} > \theta_4.$$

Contradiction. Therefore, $d_{\mathrm{Haus}}(uA, vA) \leqslant \theta_2$. This proves the claim. In particular, $d_{\mathrm{Haus}}(uA, vA) \leqslant \varepsilon$. Since $y \in A^{+\varepsilon}$, we have $uf_uy \in uA^{+\varepsilon}$ and $vf_vy \in vA^{+\varepsilon}$. Since $uf_uy, vf_vy \in uf_uY$, we have $|uf_uy - vf_vy|_{uA} \leqslant \mathrm{diam}_{uA}(uf_uY) \leqslant \varepsilon$. According to Lemma 1.7.5, $|uf_uy - vf_vy| \leqslant \theta$. This proves (CQ2). □

CHAPTER 2

UNIFORM UNIFORM EXPONENTIAL GROWTH IN SMALL CANCELLATION GROUPS

> Words are pale shadows of forgotten names. As names have power, words have power. Words can light fires in the minds of men. Words can wring tears from the hardest hearts. There are seven words that will make a person love you. There are ten words that will break a strong man's will. But a word is nothing but a painting of a fire. A name is the fire itself.
>
> from *The Name of the Wind*, of Patrick Rothfus

Contents

The results of this chapter correspond to the following article:

— X. Legaspi and M. Steenbock. Uniform uniform exponential growth in small cancellation groups, 2023. URL: `https://orcid.org/0000-0002-1497-6448`.

In Section 2.1 we will overview Gromov hyperbolic spaces, acylindricity and geometric small cancellation theory. In Section 2.2 we will see that reduced subsets generate free subgroups with the Geodesic Extension Property. This property will be relevant to the counting argument of Section 2.4.2. In Section 2.3 we generalise work of M. Koubi [56] and G. Arzhantseva - I. Lysenok, [9]. The goal is to produce reduced subsets inside uniform powers of other subsets of isometries. In Section 2.4 we study the subsets of shortening-free words of a free subgroup generated by a reduced subset. These are infinite subsets, each depending on a geometric small cancellation family, such that (i) their elements are not killed when taking the geometric small cancellation quotient and (ii) their relative growth rate does not decrease too much when taking the geometric small cancellation quotient. We will prove (i) and (ii) in Section 2.4.2 and Section 2.4.3, respectively. Finally, Section 2.5 is devoted to the proof of our main theorem (Theorem 0.6.2).

2.1 Hyperbolic geometry

We collect some facts on hyperbolic geometry in the sense of Gromov, [49], including its version of small cancellation theory, [50, 41]. See also [26, 46, 52, 32].

2.1.1 Hyperbolicity

Let X be a metric space. The *Gromov product* of three points x, y, $z \in X$ is defined by

$$(x, y)_z = \frac{1}{2}\{|x - z| + |y - z| - |x - y|\}.$$

Definition 2.1.1. — Let $\delta \geqslant 0$. The metric space X is *δ-hyperbolic* if it is geodesic and for every x, y, z and $t \in X$, the *four point inequality* holds, that is

$$(x, z)_t \geqslant \min \{(x, y)_t, (y, z)_t\} - \delta.$$

Convention 2.1.2. — Let $\delta \geqslant 0$. For the remainder of this section, we assume that the space X is δ-hyperbolic. If $\delta = 0$, then it can be isometrically embedded in an $\mathbf{R}$-tree, [46, Chapitre 2, Proposition 6]. Note that X is δ'-hyperbolic for every $\delta' \geqslant \delta$. *In this chapter we always assume for convenience that the hyperbolicity constant δ is positive.*

We write ∂X for the Gromov boundary of X. We can use the boundary defined with sequences converging at infinity, [26, Chapitre 2, Définition 1.1]. Note that we did not assume the space X to be proper, thus we use the boundary defined with sequences converging at infinity, [26, Chapitre 2, Définition 1.1]. Hyperbolicity has the following consequences.

Lemma 2.1.3 ([42, Lemmas 2.3 and 2.4])**. —** *Let $x, y, z \in X$. Then*

$$(x, y)_z \leqslant d(z, [x, y]) \leqslant (x, y)_z + 4\delta.$$

Lemma 2.1.4 ([9, Lemma 2])**. —** *Let $i \in [\![1, 2]\!]$. Let x_i, $y_i \in X$. Then*

$$|x_1 - y_1| + |x_2 - y_2| \leqslant |x_1 - x_2| + |y_1 - y_2| + 2\,\mathrm{diam}([x_1, y_1]^{+8\delta} \cap [x_2, y_2]^{+8\delta}).$$

2.1.2 Quasi-convexity

Let $\eta \geqslant 0$. A subset $Y \subset X$ is *η-quasi-convex* if every geodesic joining two points of Y is contained in $Y^{+\eta}$. For instance, geodesics are 2δ-quasi-convex. A subset $Y \subset X$ is *strongly quasi-convex* if it is 2δ-quasi-convex and for every $y, y' \in Y$, the induced path metric $|\cdot|_Y$ on Y satisfies

$$|y - y'|_X \leqslant |y - y'|_Y \leqslant |y - y'|_X + 8\delta.$$

Quasi-convexity in hyperbolic spaces has the following consequences.

Lemma 2.1.5 ([26, Chapitre 1, Proposition 3.1];[42, Lemma 2.4])**. —** *Let $\eta \geqslant 0$. Let*

$Y \subset X$ be an η-quasi-convex subset. Then for every $x \in X$ and for every y, $y' \in Y$,

$$d(x, Y) \leqslant (y, y')_x + \eta + 3\delta.$$

Given a point $x \in X$ and a subset $Y \subset X$, then $y \in Y$ is a *projection* of x on Y if

$$|x - y| \leqslant d(x, Y) + \delta.$$

Lemma 2.1.6 ([26, Chapitre 2, Proposition 2.1];[30, Lemma 2.12])**. —** *Let $\eta \geqslant 0$. Let $Y \subset X$ be an η-quasi-convex subset.*

(i) Let $x \in X$. Let y be a projection of x on Y. Then for every $y' \in Y$, $(x, y')_y \leqslant \eta + \delta$.

(ii) Let $i \in [\![1, 2]\!]$. Let $x_i \in X$. Let y_i be a projection of x_i on Y. Then,

$$|y_1 - y_2| \leqslant \max\{|x_1 - x_2| - |x_1 - y_1| - |x_2 - y_2| + 2\varepsilon, \varepsilon\},$$

where $\varepsilon = 2\eta + 3\delta$.

Lemma 2.1.7 ([26, Chapitre 10, Proposition 1.2]; [30, Lemma 2.13])**. —** *Let $\eta \geqslant 0$. Let $Y \subset X$ be an η-quasi-convex subset. Then for every $\varepsilon \geqslant \eta$, the subset $Y^{+\varepsilon}$ is 2δ-quasi-convex.*

Lemma 2.1.8 ([41, Lemma 2.2.2 (2)]; [30, Lemma 2.16])**. —** *Let $i \in [\![1, 2]\!]$. Let $\eta_i \geqslant 0$. Let $Y_i \subset X$ be an η_i-quasi-convex subset. Then for every $\varepsilon \geqslant 0$,*

$$\operatorname{diam}(Y_1^{+\varepsilon} \cap Y_2^{+\varepsilon}) \leqslant \operatorname{diam}(Y_1^{+\eta_1+3\delta} \cap Y_2^{+\eta_2+3\delta}) + 2\varepsilon + 4\delta.$$

2.1.3 Isometries

Let G be a group acting by isometries on X. Let $x \in X$ be a point.

Classification of isometries. Recall that an isometry $g \in G$ is either *elliptic*, i.e. the orbit $\langle g \rangle \cdot x$ is bounded, *loxodromic*, i.e. the map $\mathbf{Z} \to X$ sending m to $g^m x$ is a quasi-isometric embedding or *parabolic*, i.e. it is neither loxodromic or elliptic, [26, Chapitre 9, Théorème 2.1]. Note that these definitions do not depend on the point x.

Translation lengths. To measure the action of an isometry $g \in G$ on X we define the *translation length* and the *stable translation length* as

$$\|g\| = \inf_{x \in X} |gx - x|, \quad \text{and} \quad \|g\|^{\infty} = \lim_{n \to +\infty} \frac{1}{n} |g^n x - x|.$$

Note that the definition of $\|g\|^{\infty}$ does not depend on the point x. These two lengths are related as follows, [26, Chapitre 10, Proposition 6.4].

$$\|g\|^{\infty} \leqslant \|g\| \leqslant \|g\|^{\infty} + 16\delta. \tag{2.1.1}$$

The isometry g is *loxodromic* if, and only if, its stable translation length is positive, [26, Ch. 10, Prop. 6.3].

Axis. The *axis* of $g \in G$ is the set

$$A_g = \{\, x \in X \; : \; |gx - x| \leqslant \|g\| + 8\delta \,\}.$$

Lemma 2.1.9 ([41, Proposition 2.3.3];[30, Proposition 2.28]). — *Let $g \in G$. Then A_g is 10δ-quasi-convex and $\langle g \rangle$-invariant. Moreover, for every $x \in X$,*

$$\|g\| + 2d(x, A_g) - 10\delta \leqslant |gx - x| \leqslant \|g\| + 2d(x, A_g) + 10\delta.$$

ℓ^{∞}-Energy. To measure the action of a finite subset of isometries $U \subset G$ on X we define the ℓ^{∞}-energy of U at x and the ℓ^{∞}-energy of U as

$$\mathrm{L}(U, x) = \max_{u \in U} |ux - x|, \quad \text{and} \quad \mathrm{L}(U) = \inf_{x \in X} \mathrm{L}(U, x).$$

The point x is *almost-minimizing the ℓ^{∞}-energy of U* if $\mathrm{L}(U, x) \leqslant \mathrm{L}(U) + \delta$. It is easy to see that the translation length and the ℓ^{∞}-energy are related as follows. For every $g \in U$,

$$\|g\| \leqslant \mathrm{L}(U). \tag{2.1.2}$$

2.1.4 Group action on a δ-hyperbolic space

Let G be a group acting by isometries on X.

Classification of group actions. We denote by ∂G the set of all accumulation points of an orbit $G \cdot x$ in the boundary ∂X. This set does not depend on the point x. One says that the action of G on X is

- *elliptic*, if ∂G is empty, or equivalently if one (hence any) orbit of G is bounded;
- *parabolic*, if ∂G contains exactly one point;
- *loxodromic*, if ∂G contains exactly two points;
- *non-elementary*, if ∂G contains at least 3 points, or equivalently if ∂G is infinite.

If the action of G is elliptic, parabolic or loxodromic, we will say that this action is *elementary*. In this context, being elliptic (respectively parabolic, loxodromic, etc) refers to the action of G on X. However, if there is no ambiguity we will simply say that G is elliptic (respectively parabolic, loxodromic, etc).

Lemma 2.1.10 ([31, Propositon 3.6]). — *If $|\partial G| \geqslant 2$, then G contains a loxodromic isometry.*

Acylindricity. For our purpose we require some properness for this action. We will use an acylindrical action on a metric space, keeping in mind the parameters that appear in the definition, [38, Proposition 5.31]. Recall that we assumed X to be δ-hyperbolic, with $\delta > 0$.

Definition 2.1.11 (Acylindrical action). — Let κ, $N > 0$. The group G acts (κ, N)-*acylindrically on the δ-hyperbolic space X* if the following holds: for every $x, y \in X$ with $|x - y| \geqslant \kappa$, the number of elements $u \in G$ satisfying $|ux - x| \leqslant 100\delta$ and $|uy - y| \leqslant 100\delta$ is bounded above by N.

Definition 2.1.12 (Global injectivity radius). — The *global injectivity radius* of the action of G on X is

$$\mathrm{T}(G, X) = \inf\{\, \|g\|^{\infty} \ : \ g \in G \text{ loxodromic}\,\},$$

with the convention $\inf \varnothing = +\infty$.

Lemma 2.1.13 ([18, Lemma 4.2]; c.f. [35, Lemma 3.9]). — *Assume that the action of G on X is (κ, N)-acylindrical. Then*

$$\mathrm{T}(G, X) \geqslant \frac{\delta}{N}.$$

Loxodromic subgroups. Let $H \leqslant G$ be a loxodromic subgroup with limit set $\partial H = \{\xi, \eta\}$. The *$H$-invariant cylinder*, denoted by C_H, is the open 20δ-neighborhood of all $10^3\delta$-local $(1, \delta)$-quasi-geodesics with endpoints ξ and η at infinity.

Lemma 2.1.14 (Invariant cylinder; [31, Lemma 3.13])**. —** *Let $H \leqslant G$ be a loxodromic subgroup. Then the subset C_H is invariant under the action of H and strongly quasiconvex.*

Lemma 2.1.15 ([30, Corollary 2.7])**. —** *Let $\gamma\colon I \to X$ be a $10^3\delta$-local $(1, \delta)$-quasi-geodesic. Then:*

(i) For every $t, t', s \in I$ such that $t \leqslant s \leqslant t'$, we have $(\gamma(t), \gamma(t'))_{\gamma(s)} \leqslant 6\delta$.

(ii) For every $x \in X$ and for every $y, y' \in \gamma$, we have $d(x, \gamma) \leqslant (y, y')_x + 9\delta$.

The *maximal loxodromic subgroup containing* H is the stabiliser of the set ∂H. For a loxodromic element $g \in G$, we denote by $E(g)$ the *maximal loxodromic subgroup containing* g. We define the *equivalence relation* $\sim_g$ on G by $u \sim_g v$ if and only if $u^{-1}v \in E(g)$, for every $u, v \in G$. The *fellow travelling constant of a loxodromic element* $g \in G$ is

$$\Delta(g) = \sup\{\, \mathrm{diam}(uA_g^{+20\delta} \cap vA_g^{+20\delta}) \;:\; u, v \in G, u \not\sim_g v \,\}.$$

Lemma 2.1.16 ([38, Proof of Proposition 6.29])**. —** *Assume that the action of G on X is (κ, N)-acylindrical. Let $g \in G$ be a loxodromic element. Then*

$$\Delta(g) \leqslant \kappa + (N + 2)\|g\|^\infty + 100\delta.$$

Lemma 2.1.17 ([38, Lemma 6.5])**. —** *Assume that the action of G on X is acylindrical. Let $g \in G$ be a loxodromic element. Then $E(g)$ is virtually cyclic.*

The subgroup $H^+ \leqslant G$ fixing pointwise ∂H is an at most index 2 subgroup of H. The next corollary is a well-known consequence of Lemma 2.1.10, Lemma 2.1.17 and [77, Lemma 4.1].

Corollary 2.1.18. — *Assume that the action of G on X is acylindrical. The set F of all elements of finite order of H^+ is a finite normal subgroup of H. Moreover there exists a loxodromic element $h \in H^+$ such that the map $F \rtimes_\phi \langle h \rangle \to H^+$ that sends (f, g) to fg is an isomorphism, where $\phi\colon \langle h \rangle \to \mathrm{Aut}(F)$ is the action by conjugacy of $\langle h \rangle$ on F.*

For a loxodromic element $g \in G$, we denote by $F(g)$ the set of all elements of finite order of $E^+(g)$. We say that g is *primitive* if its image in $E^+(g)/F(g)$ generates the quotient.

The following lemma permits to produce primitive loxodromic elements uniformly. It will be useful during section section 2.3.

Lemma 2.1.19 ([56]; [9]; [45, Lemma 2.7]). — *For every $\kappa > 0$ and $N > 0$ there exists a positive integer n_0 with the following property. Let $U \subset G$ be a finite symmetric subset containing the identity. Assume that the action of G on X is (κ, N)-acylindrical. If $\mathrm{L}(U) > 50\delta$, then there exist a primitive loxodromic element $g \in U^{n_0}$ such that*

$$\|g\|^{\infty} \geqslant \frac{1}{2}\,\mathrm{L}(U).$$

Definition 2.1.20 (Loxodromic wideness). — The *loxodromic wideness* of the action of G on X is

$$\Phi(G, X) = \sup\{\, |F(g)| \;:\; g \in G \textit{ loxodromic} \,\},$$

with the convention $\sup \varnothing = -\infty$.

Lemma 2.1.21 ([66, Lem. 6.8]). — *Assume that the action of G on X is (κ, N)-acylindrical. Then*

$$\Phi(G, X) \leqslant N.$$

Classification of acylindrical actions. Following the proof of D. Osin [66, Theorem 1.1], one gets the following classification. It already appears in [49].

Lemma 2.1.22. — *Assume that the action of G on X is acylindrical. Then G satisfies exactly one of the following three conditions.*

(i) G is elliptic, or equivalently one (hence any) orbit of G is bounded.

(ii) G is loxodromic, or equivalently G is virtually cyclic and contains a loxodromic element.

(iii) G is non-elementary, or equivalently H contains a free group $\mathbf{F}_2$ of rank 2 and one (hence any) orbit of $\mathbf{F}_2$ is unbounded.

In particular, if the action of G on X is acylindrical, then every isometry $g \in G$ is either elliptic or loxodromic, [18]. The following trichotomy is a direct consequence of the previous lemma and [19, Theorem 13.1].

Lemma 2.1.23. — *Let G be a group acting acylindrically on a δ-hyperbolic space X. Let $U \subset G$ be a finite symmetric subset containing the identity. Then one of the following conditions holds:*

(T'1) $\mathrm{L}(U) \leqslant 10^4\delta$.

(T'2) The subgroup $\langle U\rangle$ is virtually cyclic and contains a loxodromic element.

(T'3) $\omega(U) \geqslant \frac{1}{10^3}\log 3$.

2.1.5 Small cancellation theory

Let G be a group acting by isometries on X. We recall that X is a δ-hyperbolic space.

Loxodromic moving family. The following definition generalises the conjugacy closure of a symmetrised set of relations in classical small cancellation theory.

Definition 2.1.24 (Loxodromic moving family)**. —** A *loxodromic moving family* $\mathscr{Q}$ is a set of the form

$$\mathscr{Q} = \{\, (g\,\langle h\rangle\, g^{-1}, gC_h) \in \mathscr{Q} \;:\; g \in G,\ h \in \mathscr{L}\,\},$$

where $\mathscr{L} \subset G$ is a set of loxodromic elements and C_h stands for the $\langle h\rangle$-invariant cylinder.

Let $\mathscr{Q}$ be a loxodromic moving family. The *fellow travelling constant of* $\mathscr{Q}$ is

$$\Delta(\mathscr{Q}, X) = \sup\{\, \mathrm{diam}(Y_1^{+20\delta} \cap Y_2^{+20\delta}) \;:\; (H_1, Y_1) \neq (H_2, Y_2) \in \mathscr{Q}\,\}.$$

The *injectivity radius of* $\mathscr{Q}$ is

$$\mathrm{T}(\mathscr{Q}, X) = \inf\{\, \|h\| \;:\; h \in H - \{1\}, (H, Y) \in \mathscr{Q}\,\}.$$

Note that here we require the translation length and not the stable translation length, which was present in the definition of the global injectivity radius $\mathrm{T}(G, X)$. We denote $K = \langle\langle H \mid (H, Y) \in \mathscr{Q}\rangle\rangle$ and $\bar{G} = G/K$. We denote by $\pi\colon G \twoheadrightarrow \bar{G}$ the natural projection and write $\bar{g}$ for $\pi(g)$ for short, for every $g \in G$. The notation $\bar{U}$ may refer to either a subset of $\bar{G}$ or to $\pi(U)$, for some $U \subset G$.

Definition 2.1.25 (Small cancellation condition)**. —** Let $\lambda > 0$ and $\varepsilon > 0$. We say that $\mathscr{Q}$ satisfies the *geometric $C''(\lambda, \varepsilon)$-small cancellation condition* if:

(SC1) $\Delta(\mathscr{Q}, X) < \lambda\,\mathrm{T}(\mathscr{Q}, X)$,

(SC2) $\mathrm{T}(\mathscr{Q}, X) > \varepsilon\delta$.

In that case we say that $\bar{G}$ is a *geometric $C''(\lambda, \varepsilon)$-small cancellation quotient.*

Cone-off space. Let $\rho > 0$. We denote by $\mathscr{Y}$ the collection of cylinders gC_h such that $g \in G$ and $h \in \mathscr{L}$. Let $Y \in \mathscr{Y}$. Note that $gC_h = C_{ghg^{-1}}$. The *cone of radius ρ over Y*, denoted by $Z_\rho(Y)$, is the quotient of $Y \times [0, \rho]$ by the equivalence relation that identifies all the points of the form $(y, 0)$. The *apex of the cone $Z_\rho(Y)$* is the equivalence class of $(y, 0)$. By abuse of notation, we still write $(y, 0)$ for the equivalence class of $(y, 0)$. We denote by $\mathscr{V}$ the collection of apices of the cones over the elements of $\mathscr{Y}$. Let $\iota \colon Y \hookrightarrow Z_\rho(Y)$ be the map that sends y to (y, ρ). The *cone-off space of radius ρ over X relative to $\mathscr{Q}$*, denoted by $\dot{X}_\rho = \dot{X}_\rho(\mathscr{Q}, X)$, is the space obtained by attaching for every $Y \in \mathscr{Y}$, the cone $Z_\rho(Y)$ on X along Y according to $\iota \colon Y \hookrightarrow Z_\rho(Y)$. There is a natural metric on $\dot{X}_\rho(\mathscr{Q})$ and an action by isometries of G on $\dot{X}_\rho$.

Quotient space. The *quotient space of radius ρ over X relative to $\mathscr{Q}$*, denoted by $\bar{X}_\rho = \bar{X}_\rho(\mathscr{Q}, X)$, is the orbit space $\dot{X}_\rho / K$. We denote by $\zeta \colon \dot{X}_\rho \twoheadrightarrow \bar{X}_\rho$ the natural projection and write $\bar{x}$ for $\zeta(x)$ for short. Furthermore, we denote by $\bar{\mathscr{V}}$ the image in $\bar{X}_\rho$ of the apices $\mathscr{V}$. We consider $\bar{X}_\rho$ as a metric space equipped with the quotient metric, that is for every $x, x' \in \dot{X}_\rho$

$$|\bar{x} - \bar{x}'|_{\bar{X}} = \inf_{h \in K} |hx - x'|_{\dot{X}}.$$

We note that the action of G on $\dot{X}_\rho$ induces an action by isometries of $\bar{G}$ on $\bar{X}_\rho$.

Convention 2.1.26. — In what follows, we are going to assume that X is a metric graph whose edges all have the same constant length. This is to ensure that both the cone-off space $\dot{X}_\rho$ and the quotient space $\bar{X}_\rho$ are geodesic spaces, [20, I.7.19]. This is not a restrictive assumption, as explained in [38, Section 5.3].

The following lemma summarises Proposition 3.15 and Theorem 6.11 of [30]. It will be central in the proof of Theorem 0.6.2.

Lemma 2.1.27 (Small Cancellation Theorem [30])**. —** *There exist positive numbers δ_0, $\bar{\delta}$, Δ_0, ρ_0 satisfying the following. Let $0 < \delta \leqslant \delta_0$ and $\rho > \rho_0$. Let G be a group acting by isometries on a δ-hyperbolic space X. Let $\mathscr{Q}$ be a loxodromic moving family such that $\Delta(\mathscr{Q}, X) \leqslant \Delta_0$ and $\mathrm{T}(\mathscr{Q}, X) > 100\pi \sinh \rho$. Then:*

(i) $\bar{X}_\rho$ is a $\bar{\delta}$-hyperbolic space on which $\bar{G}$ acts by isometries.

(ii) Let $r \in (0, \rho/20]$. If for all $v \in \mathscr{V}$, the distance $|x - v| \geqslant 2r$ then the projection $\zeta \colon \dot{X}_\rho \to \bar{X}_\rho$ induces an isometry from $B(x, r)$ onto $B(\bar{x}, r)$.

(iii) *Let* $(H, Y) \in \mathcal{Q}$*. If* $v \in \mathcal{V}$ *stands for the apex of the cone* $Z_\rho(Y)$*, then the natural projection* $\pi\colon G \twoheadrightarrow \bar{G}$ *induces an isomorphism from* $\operatorname{Stab}(Y)/H$ *onto* $\operatorname{Stab}(\bar{v})$*.* □

Remark 2.1.28. — It is important to note that in this statement the constants δ_0, $\bar{\delta}$, Δ_0, ρ_0 are independent of G, X, $\mathcal{Q}$ or δ. Moreover δ_0 and Δ_0 (respectively ρ_0) can be chosen arbitrarily small (respectively large). We will refer to δ_0, $\bar{\delta}$, Δ_0, ρ_0 as *the constants of the Small Cancellation Theorem*.

For the remainder of this subsection, we choose δ, ρ, G, X, and $\mathcal{Q}$ satisfying the hypothesis of the Small Cancellation Theorem (Lemma 2.1.27). The following lemmas are consequence of the Small Cancellation Theorem.

Lemma 2.1.29 ([31, Proposition 5.16]). — *Let* E *be an elliptic (respectively loxodromic) subgroup of* G *for its action on* X*. Then the image of* E *through the natural projection* $\pi\colon G \twoheadrightarrow \bar{G}$ *is elliptic (respectively elementary) for its action on* $\bar{X}_\rho$*.*

Lemma 2.1.30 ([31, Proposition 5.17]). — *Let* E *be an elliptic subgroup of* G *for its action on* X*. Then the natural projection* $\pi\colon G \twoheadrightarrow \bar{G}$ *induces an isomorphism from* E *onto its image.*

Lemma 2.1.31 ([31, Proposition 5.18]). — *Let* $\bar{E}$ *be an elliptic subgroup of* $\bar{G}$ *for its action on* $\bar{X}_\rho$*. One of the following holds.*

(i) *There exists an elliptic subgroup* E *of* G *for its action on* X *such that the natural projection* $\pi\colon G \twoheadrightarrow \bar{G}$ *induces an isomorphism from* E *onto* $\bar{E}$*.*

(ii) *There exists* $\bar{v} \in \bar{\mathcal{V}}$ *such that* $\bar{E} \subset \operatorname{Stab}(\bar{v})$*.*

Lemma 2.1.32 ([36, Proposition 9.13]). — *Let* $\bar{U} \subset \bar{G}$ *be a finite set such that* $\mathrm{L}(\bar{U}) \leqslant \rho/5$*. If, for every* $\bar{v} \in \bar{\mathcal{V}}$*, the set* $\bar{U}$ *is not contained in* $\operatorname{Stab}(\bar{v})$*, then there exists a pre-image* $U \subset G$ *of* $\bar{U}$ *of energy* $\mathrm{L}(U) \leqslant \pi \sinh \mathrm{L}(\bar{U})$*.*

Lemma 2.1.33 (Greendlinger's Lemma). — *Let* $x \in X$*. Let* $g \in G$*. If* $g \in K - \{1\}$*, then there exists* $(H, Y) \in \mathcal{Q}$ *with the following property. Let* y_0 *an* y_1 *be the respective projections of* x *and* gx *on* Y*. Then*

$$|y_0 - y_1| > \mathrm{T}(H, X) - 2\pi \sinh \rho - 23\delta.$$

Remark 2.1.34. — The previous statement is obtained from [33, Theorem 3.5] after applying [33, Proposition 1.11], [30, Proposition 2.4 (2)] and [30, Lemma 2.31]. Note that in [33, Theorem 3.5] there is an extra assumption saying that the loxodromic moving family is finite up to conjugacy. That assumption is only needed to make sure that the action is co-compact, hence the quotient group hyperbolic. We don't need it here.

Lemma 2.1.35 ([38, Proposition 5.33]). — *If the action of G on X is acylindrical, then so is the action of $\bar{G}$ on $\bar{X}_\rho$.*

2.2 Reduced subsets

Let $\delta \geqslant 0$. In this section, we fix a group G acting by isometries on a δ-hyperbolic space X. The *set of the inverses in G of the elements of $U \subset G$* is represented by U^{-1}.

Definition 2.2.1. — Let $\alpha > 0$. We say that a finite subset $U \subset G$ is *α-reduced at $p \in X$* if $U \cap U^{-1} = \varnothing$ and for every pair of distinct $u_1, u_2 \in U \sqcup U^{-1}$,

$$(u_1p, u_2p)_p < \frac{1}{2}\min\{|u_1p - p|, |u_2p - p|\} - \alpha - 2\delta.$$

Remark 2.2.2. — If $U \subset G$ is α-reduced at $p \in X$, then $|up-p| > 2\alpha$, for every $u \in U \sqcup U^{-1}$.

We clarify some vocabulary. Let $U \subset G$ be a subset. A *letter* is an element of the *alphabet* $U \sqcup U^{-1}$. A *word over* $U \sqcup U^{-1}$ is any finite sequence $u_1 \cdots u_n$ with $u_i \in U \sqcup U^{-1}$. The number n is called the *length* of the the given word $u_1 \cdots u_n$. We denote by $|w|_U$ the length of any word w over $U \sqcup U^{-1}$. We admit the word of length 0, the *empty word*. We write $w_1 \equiv w_2$ to express letter-for-letter equality of words w_1 and w_2 over $U \sqcup U^{-1}$. A word $u_1 \cdots u_n$ over $U \sqcup U^{-1}$ is *reduced* if it does not contain a pair of adjacent letters of the form $u_iu_i^{-1}$ or $u_i^{-1}u_i$. The *free group* $\mathbf{F}(U)$ is the set of reduced words over $U \sqcup U^{-1}$ with the group operation "concatenate and reduce". The *natural homomorphism* $\psi\colon \mathbf{F}(U) \to G$ is the evaluation of the elements of $\mathbf{F}(U)$ on G.

2.2.1 Broken geodesics

The next lemma is used to produce quasi-geodesics by concatenating some sequences of points of X with geodesics.

Lemma 2.2.3 (Broken Geodesic Lemma [9, Lemma 1]). — *Let $n \geqslant 2$. Let $x_0, \cdots, x_n$ be a sequence of $n+1$ points of X. Assume that*

$$(x_{i-1}, x_{i+1})_{x_i} + (x_i, x_{i+2})_{x_{i+1}} < |x_i - x_{i+1}| - 3\delta, \tag{2.2.1}$$

for every $i \in [\![1, n-2]\!]$*. Then the following holds.*

(i) $|x_0 - x_n| \geqslant \sum_{i=0}^{n-1} |x_i - x_{i+1}| - 2\sum_{i=1}^{n-1}(x_{i-1}, x_{i+1})_{x_i} - 2(n-2)\delta.$

(ii) $(x_0, x_n)_{x_j} \leqslant (x_{j-1}, x_{j+1})_{x_j} + 2\delta$, *for every* $j \in [\![1, n-1]\!]$.

(iii) *The geodesic* $[x_0, x_n]$ *lies in the* 5δ*-neighbourhood of the broken geodesic* $\gamma = [x_0, x_1] \cup \cdots \cup [x_{n-1}, x_n]$*, while* γ *is contained in the* r*-neighbourhood of* $[x_0, x_n]$*, where*

$$r = \sup_{1 \leqslant i \leqslant n-1} (x_{i-1}, x_{i+1})_{x_i} + 14\delta.$$

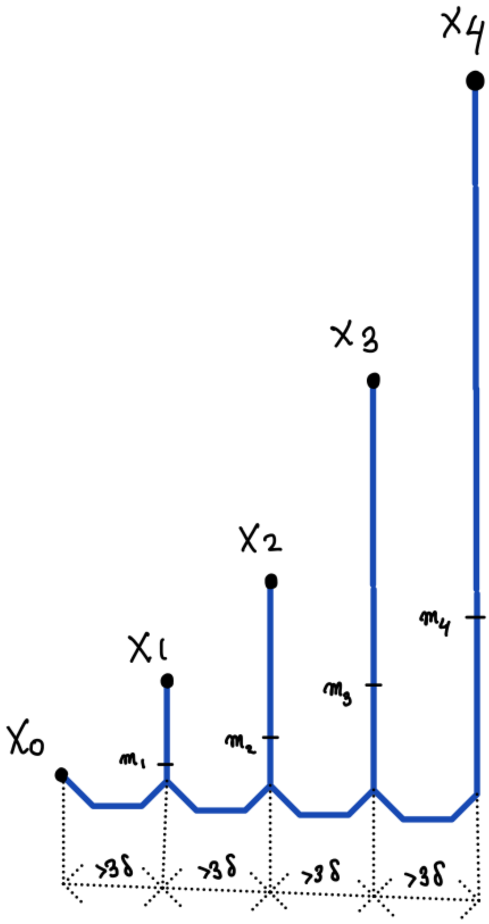

Figure 2.1 – A sequence (x_i) satisfying Equation 2.2.1. This sequence does not correspond to a reduced word over a reduced subset since for every i, the midpoint m_i of the geodesic $[x_{i-1}, x_i]$ falls inside the overlap of two consecutive geodesics.

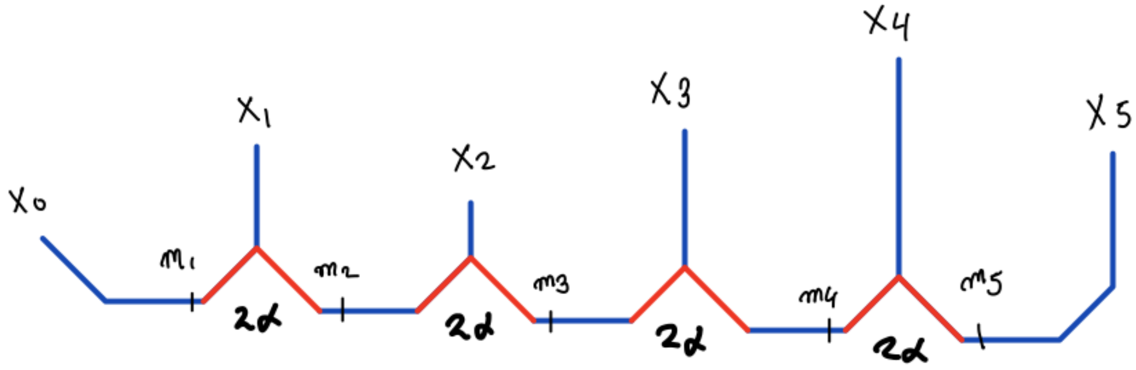

Figure 2.2 – Another sequence (x_i) satisfying Equation 2.2.1. This sequence could correspond to a reduced word over an α-reduced subset since for every i, the midpoint m_i of the geodesic $[x_{i-1}, x_i]$ falls at distance at least α from the the overlap of two consecutive geodesics. The geodesic segments in red have length 2α. In particular, every geodesic $[x_{i-1}, x_i]$ that does not fall in any of the two extremes has length at least 2α.

We verify the condition of Lemma 2.2.3 permitting to obtain broken geodesics.

Proposition 2.2.4. — *Let $\alpha > 0$. Let $U \subset G$ be an α-reduced subset at $p \in X$. Let $n \geqslant 2$. Let $w \equiv u_1 \cdots u_n$ be an element of $\mathbf{F}(U)$. Consider the sequence of $n+1$ points*

$$x_0 = p, \quad x_1 = u_1 p, \quad x_2 = u_1 u_2 p, \quad \cdots, \quad x_n = u_1 \cdots u_n p.$$

Then

(i) $(x_{i-1}, x_{i+1})_{x_i} + (x_i, x_{i+2})_{x_{i+1}} < |x_i - x_{i+1}| - 2(\alpha + 2\delta)$, *for every* $i \in [\![1, n-2]\!]$.

(ii) $|wp - p| \geqslant \frac{1}{2}|u_1 p - p| + \frac{1}{2}|u_n p - p| + 2(n-1)(\alpha + \delta) + 2\delta$.

Proof. — (i) Let $i \in [\![1, n-2]\!]$. We have

$$(x_{i-1}, x_{i+1})_{x_i} = (u_i^{-1} p, u_{i+1} p)_p, \quad (x_i, x_{i+2})_{x_{i+1}} = (u_{i+1}^{-1} p, u_{i+2} p)_p$$

and $|x_i - x_{i+1}| = |p - u_{i+1} p|$. Since w is a reduced word over $U \sqcup U^{-1}$, we have $u_i^{-1} \neq u_{i+1}$ and $u_{i+1}^{-1} \neq u_{i+2}$. Hence we can apply the fact that the subset U is α-reduced at p, obtaining

$$(u_i^{-1} p, u_{i+1} p)_p < \frac{1}{2}|u_{i+1} p - p| - \alpha - 2\delta, \quad (u_{i+1}^{-1} p, u_{i+2} p)_p < \frac{1}{2}|u_{i+1}^{-1} p - p| - \alpha - 2\delta.$$

It remains to add the two above inequalities to obtain

$$(x_{i-1}, x_{i+1})_{x_i} + (x_i, x_{i+2})_{x_{i+1}} < |x_i - x_{i+1}| - 2(\alpha + 2\delta).$$

(ii) Since $n \geqslant 2$, applying (i) and Lemma 2.2.3 (i) to the sequence $x_0, \cdots, x_n$, we obtain

$$\begin{aligned} |wp - p| \geqslant\ & |u_1 p - p| + \sum_{i=2}^{n-1} |u_i p - p| + |u_n p - p| \\ & - (u_1^{-1} p, u_2 p)_p - \sum_{i=2}^{n-1} [(u_i^{-1} p, u_{i+1} p)_p + (u_{i-1}^{-1} p, u_i p)_p] - (u_{n-1}^{-1} p, u_n p) \\ & - 2(n-2)\delta. \end{aligned}$$

Since U is α-reduced at p,

$$\sum_{i=2}^{n-1} [(u_i^{-1} p, u_{i+1} p)_p + (u_{i-1}^{-1} p, u_i p)_p] < \sum_{i=2}^{n-1} |u_i p - p| - 2(n-2)(\alpha + 2\delta).$$

and

$$(u_1^{-1} p, u_2 p)_p < \frac{1}{2} |u_1 p - p| - \alpha - 2\delta, \quad (u_{n-1}^{-1} p, u_n p) < \frac{1}{2} |u_n p - p| - \alpha - 2\delta.$$

Consequently,

$$|wp - p| \geqslant \frac{1}{2} |u_1 p - p| + \frac{1}{2} |u_n p - p| + 2(n-1)(\alpha + \delta) + 2\delta.$$

□

2.2.2 Quasi-isometric embedding of a free group

Recall that $\mathrm{L}(U, p)$ denotes the ℓ^∞-energy of $U \subset G$ at $p \in X$ (subsection 2.1.3).

Proposition 2.2.5. — *Let $\alpha > 0$. Let $U \subset G$ be an α-reduced subset at $p \in X$. Then, for every $w \in \mathbf{F}(U)$, we have*

$$2\alpha |w|_U \leqslant |wp - p| \leqslant \mathrm{L}(U, p) |w|_U.$$

In particular, the natural homomorphism $\psi \colon \mathbf{F}(U) \to G$ is injective.

Proof. — Let $w \equiv u_1 \cdots u_n$ be an element of $\mathbf{F}(U)$. If $n = 0$, then there is nothing to do. If $n = 1$, then the result is a direct consequence of the fact that the subset U is α-reduced. Assume that $n \geqslant 2$. It follows from the triangle inequality that $|wp - p| \leqslant \mathrm{L}(U, p)n$. In regards to the second inequality, we apply Proposition 2.2.4 (ii) to the sequence of $n + 1$ points

$$x_0 = p, \quad x_1 = u_1, \quad x_2 = u_1 u_2 p, \quad \cdots, \quad x_n = wp = u_1 \cdots u_n p,$$

to obtain

$$|wp - p| \geqslant \frac{1}{2}|u_1 p - p| + \frac{1}{2}|u_n p - p| + 2(n-1)(\alpha + \delta) + 2\delta.$$

According to Remark 2.2.2, we have

$$\max \{|u_1 p - p|, |u_n p - p|\} \geqslant 2\alpha.$$

Hence,

$$|wp - p| \geqslant 2\alpha n.$$

Finally, if $w \in \mathbf{F}(U)$ is not the empty word, then $|wp - p| \geqslant 2\alpha$. By definition, $\alpha > 0$. Therefore $w \neq 1$ in G. Consequently, the natural homomorphism $\psi\colon \mathbf{F}(U) \to G$ is injective. ☐

2.2.3 Geodesic extension property

This is the main result of this section. Our proof is based on [36, Lemma 3.2].

Proposition 2.2.6. — *Let $\alpha > 0$. Let $U \subset G$ be an α-reduced subset at p. Let $w \equiv u_1 \cdots u_m$ and $w' \equiv u'_1 \cdots u'_{m'}$ be two elements of $\mathbf{F}(U)$. Then U satisfies the geodesic extension property, that is, if*

$$(p, w'p)_{wp} < \frac{1}{2}|u_m p - p| - \delta,$$

then w is a prefix of w'.

Remark 2.2.7. — The *geodesic extension property* has the following meaning: if the geodesic $[p, w'p]$ extends $[p, wp]$ as a path in X, then w' extends w as a word over $U \sqcup U^{-1}$.

Proof. — The proof is by contrapositive. Assume that w is not a prefix of w'. Let r be the largest integer such that $u_i = u'_i$, for every $i \in [\![1, r-1]\!]$. In particular, $r \in [\![1, m]\!]$. For simplicity, denote

$$q = u_1 \cdots u_{r-1} p = u'_1 \cdots u'_{r-1} p.$$

It follows from the four point inequality that

$$(p, w'p)_{wp} \geqslant \min\{(p,q)_{wp}, (q, wp')_{wp}\} - \delta. \tag{2.2.2}$$

From now on, the focus will be on showing that

$$\min\{(p,q)_{wp}, (q, wp')_{wp}\} \geqslant \frac{1}{2}|u_m p - p|.$$

Using the definition of Gromov product,

$$(p,q)_{wp} = |wp - q| - (p, wp)_q, \quad (q, w'p)_{wp} = |wp - q| - (wp, w'p)_q. \tag{2.2.3}$$

We are going to estimate $|wp - q|$, $(p, wp)_q$, and $(wp, w'p)_q$.

Claim 2.2.8. — $|wp - q| \geqslant \frac{1}{2}|u_r p - p| + \frac{1}{2}|u_m p - p| + 2(m-r)(\alpha+\delta)$.

Proof. — Note that $m - r + 1 \geqslant 1$. If $m - r + 1 = 1$, then there is nothing to do. If $m - r + 1 \geqslant 2$, then we apply Proposition 2.2.4 (ii) to the sequence of $m - r + 2$ points

$$q = u_1 \cdots u_{r-1} p, \quad u_1 \cdots u_r p, \quad u_1 \cdots u_{r+1} p, \quad \cdots, \quad wp = u_1 \cdots u_m p,$$

and we obtain

$$|wp - q| \geqslant \frac{1}{2}|u_r p - p| + \frac{1}{2}|u_m p - p| + 2(m-r)(\alpha+\delta).$$

□

For simplicity, denote

$$t = u_1 \cdots u_r p \quad \text{and} \quad t' = u'_1 \cdots u'_r p.$$

Claim 2.2.9. — $(p, wp)_q < \frac{1}{2}|u_r p - p|$.

Proof. — Applying Lemma 2.2.3 (ii) and Proposition 2.2.4 (i) to the sequence of $m+1$ points

$$p, \quad u_1 p, \quad u_1 u_2 p, \quad \cdots, \quad wp = u_1 \cdots u_m p,$$

we get

$$(p, wp)_q \leqslant (u_1 \cdots u_{r-2}p, t)_q + 2\delta.$$

Since U is α-reduced at p,

$$(u_1 \cdots u_{r-2}p, t)_q = (u_{r-1}^{-1}p, u_r p)_p < \frac{1}{2}|u_r p - p| - \alpha - 2\delta.$$

Consequently,

$$(p, wp)_q < \frac{1}{2}|u_r p - p| - \alpha.$$

This proves our claim. □

Claim 2.2.10. — $(wp, w'p)_q < \frac{1}{2}|u_r p - p|$.

Proof. — If $r - 1 = m'$, then $w'p = q$ and the claim holds. Hence we can suppose that $r - 1 < m'$. It follows from the choice of r that $u_r \neq u'_r$. It follows from the four point inequality that

$$\min\{(t, wp)_q, (wp, w'p)_q, (w'p, t')_q\} \leqslant (t, t')_q + 2\delta.$$

Since U is α-reduced at p,

$$(t, t')_q = (u_r p, u'_r p)_q < \frac{1}{2}\min\{|u_r p - p|, |u'_r p - p|\} - \alpha - 2\delta.$$

Consequently,

$$\min\{(t, wp)_q, (wp, w'p)_q, (w'p, t')_q\} < \frac{1}{2}\min\{|u_r p - p|, |u'_r p - p|\} - \alpha. \tag{2.2.4}$$

We must prove that the minimum of Equation 2.2.4 is attained by $(wp, w'p)_q$. In order to do so, let's see first that the minimum of Equation 2.2.4 is not achieved by $(t, wp)_q$. Using the definition of Gromov product,

$$(t, wp)_q = |q - t| - (q, wp)_t.$$

By definition,

$$|q - t| = |u_r p - p|.$$

Recall that $m - r + 1 \geqslant 1$. If $m - r + 1 = 1$, we have

$$(q, wp)_t = (u_r^{-1}p, p)_p = 0.$$

If $m - r + 1 \geqslant 2$, applying Lemma 2.2.3 (ii) and Proposition 2.2.4 (i) to the sequence of $m - r + 2$ points

$$q = u_1 \cdots u_{r-1} p, \quad t = u_1 \cdots u_r p, \quad u_1 \cdots u_{r+1} p, \quad \cdots, \quad wp = u_1 \cdots u_m p,$$

we obtain

$$(q, wp)_t \leqslant (q, u_1 \cdots u_{r+1} p)_t + 2\delta.$$

Since U is α-reduced,

$$(q, u_1 \cdots u_{r+1} p)_t = (u_r^{-1} p, u_{r+1} p)_p < \frac{1}{2}|u_r p - p| - \alpha - 2\delta.$$

Consequently,

$$(t, wp)_q \geqslant \frac{1}{2}|u_r p - p| > \frac{1}{2}|u_r p - p| - \alpha.$$

Thus, the minimum of Equation 2.2.4 cannot be achieved by $(t, wp)_q$. Similarly, it cannot be achieved by $(w'p, t')_q$. Therefore, the only possibility is that it is achieved by $(wp, w'p)_q$. This proves our claim. □

Finally, combining Equation 2.2.2 and Equation 2.2.3 with our three claims, we obtain

$$(p, w'p)_{wp} \geqslant \min\{(p, q)_{wp}, (q, w'p)_{wp}\} - \delta > \frac{1}{2}|u_m p - p| - \delta.$$

□

2.3 Growth in groups acting on a δ-hyperbolic space

In this section, we review and adapt some of the techniques of M. Koubi. [56] – further developed by G. Arzhantseva and I. Lysenok, [9]. These techniques permit to study exponential growth rates of finite symmetric subsets in groups acting by isometries on hyperbolic spaces in the sense of M. Gromov. In particular, we clarify what are the involved parameters for acylindrical actions, which permits to obtain Theorem 2.3.8.

2.3.1 Growth of maximal loxodromic subgroups.

Let G be a group acting acylindrically on a hyperbolic space X. The goal of this subsection is to prove that the maximal loxodromic subgroups of G have some sort of uniform linear growth. We adapt an argument that was written for hyperbolic groups in [5, p. 484]. Recall that $\Phi(G, X)$ stands by the loxodromic wideness of the action of G on X (Definition 2.1.20). Given a loxodromic element $g \in G$, we denoted by $\|g\|^\infty$ its stable translation length (subsection 2.1.3) and by $E(g)$ the maximal loxodromic subgroup of G containing g (subsection 2.1.4).

Proposition 2.3.1. — *Let G be a group acting acylindrically on a hyperbolic space X. Let $U \subset G$ be a finite symmetric subset containing the identity. Let $g \in G$ be a primitive loxodromic element. Then, for every $n \geqslant 1$,*

$$|U^n \cap E(g)| \leqslant 2\Phi(G, X)\left(\frac{\mathrm{L}(U)}{\|g\|^\infty}4n + 1\right).$$

First, we focus on the case of the cyclic group generated by a loxodromic isometry.

Lemma 2.3.2. — *Let G be a group acting acylindrically on a hyperbolic space X. Let $U \subset G$ be a finite symmetric subset containing the identity. Let $g \in G$ be a loxodromic element. Then, for every $n \geqslant 1$,*

$$|U^n \cap \langle g \rangle\,| \leqslant \frac{\mathrm{L}(U)}{\|g\|^\infty}2n + 1.$$

Proof. — Let $n \geqslant 1$. We have,

$$|U^n \cap \langle g \rangle\,| = |\{\, k \in \mathbf{Z} \,:\, g^k \in U^n \,\}|.$$

Since the subset U is symmetric,

$$|\{\, k \in \mathbf{Z} \,:\, g^k \in U^n \,\}| \leqslant 2|\{\, k \in \mathbf{N} - \{0\} \,:\, g^k \in U^n \,\}| + 1.$$

Let $k \geqslant 1$ such that $g^k \in U^n$. Since the element g is loxodromic, we have $\|g\|^\infty > 0$. Observe that

$$k = \frac{\|g^k\|^\infty}{\|g\|^\infty}.$$

Let $x \in X$. Then

$$\|g^k\|^\infty \leqslant \|g^k\| \leqslant |g^k x - x| \leqslant \max_{h \in U^n} |hx - x| = \mathrm{L}(U^n, x).$$

Since the point x is arbitrary, we get $\|g^k\|^\infty \leqslant \mathrm{L}(U^n)$. By the triangle inequality, $\mathrm{L}(U^n) \leqslant$

$n\,\mathrm{L}(U)$. Hence,

$$k \leqslant \frac{\mathrm{L}(U)}{\|g\|^{\infty}} n.$$

Therefore,

$$|U^n \cap \langle g \rangle| \leqslant \frac{\mathrm{L}(U)}{\|g\|^{\infty}} 2n + 1.$$

□

We are ready for the proof of the proposition.

Proof of Proposition 2.3.1. — Let $F(g)$ be the set of all elements of finite order of $E^+(g)$. Recall that $F(g)$ is a normal subgroup of $E^+(g)$. Since the action of G on X is acylindrical and $E(g)$ is a loxodromic subgroup of G, there exists a loxodromic element $h \in E^+(g)$ such that the map

$$F(g) \rtimes_{\phi} \langle h \rangle \to E^+(g), (f, k) \mapsto fk$$

is a group isomorphism, where $\phi\colon \langle h \rangle \to \mathrm{Aut}(F(g))$ is the action by conjugacy of $\langle h \rangle$ on $F(g)$ (Corollary 2.1.18). Let $n \geqslant 1$. Let E_0 be a set of representatives of $E(g)/\langle h \rangle$. We have

$$|U^n \cap E(g)| = \sum_{r \in E_0} |U^n \cap r \langle h \rangle|.$$

First we are going to estimate $|E_0|$. By definition, $[E(g)\colon E^+(g)] \leqslant 2$. Since the homomorphism

$$\langle h \rangle \to F(g) \rtimes_{\phi} \langle h \rangle, k \mapsto (1, k)$$

is a split of the exact sequence,

$$0 \longrightarrow F(g) \overset{\iota}{\hookrightarrow} F(g) \rtimes_{\phi} \langle h \rangle \overset{\pi}{\twoheadrightarrow} \langle h \rangle \longrightarrow 0$$

we have $[E^+(g)\colon \langle h \rangle] = |F(g)| \leqslant \Phi(G, X)$. Consequently,

$$|E_0| \leqslant 2\Phi(G, X).$$

Since the action of G on X is acylindrical, we have $\Phi(G, X) < \infty$ (Lemma 2.1.21).

Now we are going to estimate $|U^n \cap r\langle h \rangle|$ for $r \in E_0$. We may assume that $U^n \cap r\langle h \rangle$ is non-empty. Then there exist $s \in U^n \cap r\langle h \rangle$. In particular $r\langle h \rangle = s\langle h \rangle$. Hence,

$$|U^n \cap r \langle h \rangle| = |U^n \cap s \langle h \rangle| = |s(s^{-1}U^n \cap \langle h \rangle)| = |s^{-1}U^n \cap \langle h \rangle|.$$

Since U is symmetric, $s^{-1} \in U^n$. Since U contains the identity, $s^{-1}U^n \subset U^{2n}$. Therefore,

$$|s^{-1}U^n \cap \langle h \rangle| \leqslant |U^{2n} \cap \langle h \rangle|.$$

According to Lemma 2.3.2,

$$|U^{2n} \cap \langle h \rangle| \leqslant \frac{\mathrm{L}(U)}{\|h\|^{\infty}} 4n + 1.$$

Consequently,

$$|U^n \cap r \langle h \rangle| \leqslant \frac{\mathrm{L}(U)}{\|h\|^{\infty}} 4n + 1.$$

Finally, since the element g is primitive, we have that $g \in \{h, h^{-1}\}$. It follows from our two estimations above that

$$|U^n \cap E(g)| \leqslant 2\Phi(G, X) \left(\frac{\mathrm{L}(U)}{\|g\|^{\infty}} 4n + 1 \right).$$

□

Given a subset $U \subset G$ and a loxodromic element $g \in G$, we fix a set of representatives $U(g)$ of the equivalence relation induced on U by $\sim_g$. Recall that the equivalence relation $\sim_g$ on G was previously defined by $u \sim_g v$ if and only if $u^{-1}v \in E(g)$, for every $u, v \in G$ (subsection 2.1.4). The reason that makes the set $U(g)$ of interest is that the set of conjugates of g by the elements of $U(g)$ is a set of "independent" loxodromic elements and has the same size as $U(g)$. We obtain the following.

Corollary 2.3.3. — *Let G be a group acting acylindrically on a hyperbolic space X. Let $U \subset G$ be a finite symmetric subset containing the identity. Let $g \in G$ be a primitive loxodromic element. Let*

$$a_0 = 2\Phi(G, X) \left(\frac{\mathrm{L}(U)}{\|g\|^{\infty}} 8 + 1 \right).$$

Then,

$$|U(g)| \geqslant \frac{1}{a_0} |U|.$$

Proof. — Consider the surjective map $U \to U(g)$ that sends every element of U to its class representative in $U(g)$. We are going to estimate its injectivity. Let $u, v \in U$ such that $u \sim_g v$. By definition, $u^{-1}v \in E(g)$. Since the subset U is symmetric, $u^{-1}v \in U^2$. Therefore, $v \in u(U^2 \cap E(g))$. Note that $|u(U^2 \cap E(g))| = |U^2 \cap E(g)|$. Consequently, each $u \in U(g)$

has at most $|U^2 \cap E(g)|$ elements in its equivalence class. According to Proposition 2.3.1, $|U^2 \cap E(g)| \leqslant a_0$. Therefore,

$$|U(g)| \geqslant \frac{1}{a_0}|U|.$$

□

2.3.2 Producing reduced subsets

Recall that given a loxodromic element $g \in G$, we denoted by $\Delta(g)$ its fellow travelling constant (subsection 2.1.4). The goal of this subsection is to produce a reduced subset using the conjugates of a loxodromic isometry of large stable translation length. More precisely, we will prove the following.

Proposition 2.3.4. — *Let $\delta > 0$ and $\alpha > 0$. Let G be a group acting acylindrically on a δ-hyperbolic space X. Let $U \subset G$ be a finite symmetric subset containing a loxodromic element $g \in U$ such that $\|g\|^\infty > 10^3\delta$. Let $p \in X$. Let*

$$b_0 = \frac{200}{\|g\|^\infty}[\Delta(g) + \mathrm{L}(U, p) + \delta + \alpha].$$

Then for every $b \geqslant b_0$, the set $S = \{ ug^bu^{-1} \ : \ u \in U(g) \}$ satisfies the following:

(i) $S \subset U^{b+2}$.

(ii) $|S| = |U(g)|$.

(iii) S *is α-reduced at p.*

Proof. — The conclusions (i) and (ii) are immediate. We are going to prove (iii) S is α-reduced at p (Definition 2.2.1). By construction, $S \cap S^{-1} = \varnothing$. Let $i \in [\![1, 2]\!]$. Let $u_i \in U$. Let $\varepsilon_i \in \{-1, 1\}$. Assume that the elements $u_1g^{\varepsilon_1 b}u_1^{-1}$ and $u_2g^{\varepsilon_2 b}u_2^{-1}$ are distinct.

Case $u_1 = u_2$. Since the elements $u_1g^{\varepsilon_1 b}u_1^{-1}$ and $u_2g^{\varepsilon_2 b}u_2^{-1}$ are distinct, we have $\varepsilon_1 = -\varepsilon_2$. Denote $h = u_1g^{\varepsilon_1 b}u_1^{-1}$. It is enough to prove that

$$(hp, h^{-1}p)_p \leqslant \frac{b}{2}\|g\|^\infty - \alpha - 2\delta.$$

Let η^- and η^+ be the points of ∂X fixed by $\langle h\rangle$ and $\gamma\colon \mathbf{R} \to X$ be an $\langle h\rangle$-invariant $10^3\delta$-local $(1, \delta)$-quasi-geodesic joining η^- to η^+. This choice is possible since $\|g\|^\infty > 10^3\delta$. It follows from Lemma 2.1.15 applied to γ that

$$(hp, h^{-1}p)_p \leqslant \mathrm{L}(U, p) + 6\delta.$$

It is clear that

$$\mathrm{L}(U,p) + 6\delta \leqslant \frac{b}{2}\,\|g\|^{\infty} - \alpha - 2\delta.$$

‘

Case $u_1 \neq u_2$. In particular $u_1 \not\sim_g u_2$, which means that $u_1^{-1}u_2$ does not belong to $E(g)$.

Claim 2.3.5. — $d(p, A_g) \leqslant \frac{1}{2}\,\mathrm{L}(U,p) + 5\delta$.

Proof. — It follows from Lemma 2.1.9 that

$$d(p, A_g) \leqslant \frac{1}{2}|gp - p| + 5\delta.$$

Moreover, since $g \in U$, we have $|gp - p| \leqslant \mathrm{L}(U,p)$. This proves our claim.

□

Consider the points $x_i = u_i p$ and $y_i = u_i g^{\varepsilon_i b} p$.

Claim 2.3.6. — $\mathrm{diam}([x_1, y_1]^{+8\delta} \cap [x_2, y_2]^{+8\delta}) \leqslant \Delta(g) + \mathrm{L}(U,p) + 44\delta$.

Proof. — Denote $\sigma = d(p, A_g) + 10\delta$. We have,

$$\max\left\{d(x_i, u_i A_g), d(y_i, u_i A_g)\right\} \leqslant \sigma.$$

Recall that the axis A_g is 10δ-quasi-convex (Lemma 2.1.9). Hence, since $\sigma \geqslant 10\delta$, the subset $u_i A_g^{+\sigma}$ is 2δ-quasi-convex (Lemma 2.1.7). Consequently,

$$[x_i, y_i] \subset u_i A_g^{+\sigma+2\delta}.$$

Therefore,

$$\mathrm{diam}([x_1, y_1]^{+8\delta} \cap [x_2, y_2]^{+8\delta}) \leqslant \mathrm{diam}(u_1 A_g^{+\sigma+10\delta} \cap u_2 A_g^{+\sigma+10\delta}).$$

According to Lemma 2.1.8,

$$\mathrm{diam}(u_1 A_g^{+\sigma+10\delta} \cap u_2 A_g^{+\sigma+10\delta}) \leqslant \mathrm{diam}(u_1 A_g^{+13\delta} \cap u_2 A_g^{+13\delta}) + 2(\sigma + 10\delta) + 4\delta.$$

Moreover,

$$\mathrm{diam}(u_1 A_g^{+13\delta} \cap u_2 A_g^{+13\delta}) \leqslant \mathrm{diam}(u_1 A_g^{+20\delta} \cap u_2 A_g^{+20\delta}).$$

Since $u_1^{-1}u_2$ does not belong to $E(g)$,

$$\operatorname{diam}(u_1 A_g^{+20\delta} \cap u_2 A_g^{+20\delta}) \leqslant \Delta(g).$$

Since the action of G on X is acylindrical, we have $\Delta(g) < \infty$ (Lemma 2.1.16). Combining the above estimations with the previous claim, we obtain

$$\operatorname{diam}([x_1, y_1]^{+8\delta} \cap [x_2, y_2]^{+8\delta}) \leqslant \Delta(g) + \mathrm{L}(U, p) + 54\delta.$$

This proves our claim.

□

Denote $s_i = u_i g^{\varepsilon_i b} u_i^{-1}$.

Claim 2.3.7. — $(s_1 p, s_2 p)_p \leqslant \Delta(g) + 5\,\mathrm{L}(U, p) + 54\delta$.

Proof. — By definition,

$$(s_1 p, s_2 p)_p = \frac{1}{2}(|s_1 p - p| + |s_2 p - p| - |s_1 p - s_2 p|).$$

By the triangle inequality,

$$|s_i p - p| \leqslant |x_i - y_i| + 2|u_i p - p|,$$
$$|s_1 p - s_2 p| \geqslant |y_1 - y_2| - |u_1 p - p| - |u_2 p - p|.$$

Consequently,

$$(s_1 p, s_2 p)_p \leqslant \frac{1}{2}(|x_1 - y_1| + |x_2 - y_2| - |y_1 - y_2|) + \frac{3}{2}(|u_1 p - p| + |u_2 p - p|).$$

Combining the previous claim with Lemma 2.1.4, we obtain

$$|x_1 - y_1| + |x_2 - y_2| - |y_1 - y_2| \leqslant |x_1 - x_2| + 2(\Delta(g) + \mathrm{L}(U, p) + 44\delta).$$

By the triangle inequality,

$$|x_1 - x_2| \leqslant |u_1 p - p| + |u_2 p - p|.$$

Moreover, since $u_i \in U$, we have $|u_i p - p| \leqslant \mathrm{L}(U, p)$. Combining the above estimations, we obtain

$$(s_1p, s_2p)_p \leqslant \Delta(g) + 5\,\mathrm{L}(U,p) + 44\delta.$$

This proves our claim.

□

Finally, note that

$$\frac{1}{2}\min\{|s_1p - p|, |s_2p - p|\} - \alpha - 2\delta \geqslant \frac{b}{2}\,\|g\|^{\infty} - \alpha - 2\delta.$$

Since $b \geqslant b_0$, we obtain

$$\frac{b}{2}\,\|g\|^{\infty} - \alpha - 2\delta > \Delta(g) + 5\,\mathrm{L}(U,p) + 54\delta.$$

Therefore, the previous claim implies that

$$(s_1p, s_2p)_p < \frac{1}{2}\min\{|s_1p - p|, |s_2p - p|\} - \alpha - 2\delta.$$

□

2.3.3 Growth trichotomy

We are going to combine the two previous subsections in the following result.

Theorem 2.3.8 (Theorem 0.6.10)**. —** *For every $\kappa > 0$ and $N > 0$, there exist an integer $c > 1$ with the following property. Let $\delta > 0$ and $\alpha > 0$. Let G be a group acting (κ, N)-acylindrically on a δ-hyperbolic space X. Let $U \subset G$ be a finite symmetric subset containing the identity. Let $p \in X$ be a point almost-minimizing the ℓ^∞-energy $\mathrm{L}(U)$. Then one of the following conditions holds:*

(T1) $\mathrm{L}(U) \leqslant 10^4 \max\{\kappa, \delta, \alpha\}$.

(T2) The subgroup $\langle U\rangle$ is virtually cyclic and contains a loxodromic element.

(T3) There exist a finite subset $S \subset G$ with the following properties:

- *(i)* $S \subset U^c$,
- *(ii)* $|S| \geqslant \max\left\{2, \frac{1}{c}|U|\right\}$,
- *(iii) S is α-reduced at p.*

Moreover, $\omega(U) \geqslant \frac{1}{c}\log|U|$.

Proof. — Let $\kappa > 0$ and $N > 0$. Let n_0 be the positive integer of Lemma 2.1.19 depending on κ and N. We fix auxiliar parameters

$$a_1 = 200Nn_0, \quad \text{and} \quad b_1 = 200(N+2) + 500n_0 + 700.$$

We put

$$c \geqslant \max\left\{a_1, n_0(b_1+2), \frac{2n_0(b_1+2)\log a_1}{\log 2}\right\}.$$

Let $\delta > 0$ and $\alpha > 0$. Let G be a group acting (κ, N)-acylindrically on a δ-hyperbolic space X. Let $U \subset G$ be a finite symmetric subset containing the identity. Let $p \in X$ be a point almost-minimizing the ℓ^∞-energy $\mathrm{L}(U)$. Assume that $\mathrm{L}(U) > 10^4 \max\{\kappa, \delta, \alpha\}$. Since $\mathrm{L}(U) > 50\delta$, according to Lemma 2.1.19 there exist a primitive loxodromic element $g \in U^{n_0}$ such that

$$\|g\|^\infty \geqslant \frac{1}{2}\mathrm{L}(U). \tag{2.3.1}$$

In particular $\|g\|^\infty \geqslant 10^3\delta$. Let $H = \langle U \rangle$. Note that the loxodromic g belongs to H. Assume in addition that the subgroup H is not virtually cyclic. We prove (T3). We are going to apply Corollary 2.3.3 and Proposition 2.3.4 to U^{n_0} and g. Let

$$a_0 = 2\Phi(G,X)\left(\frac{\mathrm{L}(U^{n_0})}{\|g\|^\infty}8+1\right), \quad b_0 = \frac{200}{\|g\|^\infty}[\Delta(g) + \mathrm{L}(U^{n_0}, p) + \delta + \alpha].$$

By the triangle inequality,

$$\mathrm{L}(U^{n_0}) \leqslant n_0\,\mathrm{L}(U), \quad \text{and} \quad \mathrm{L}(U^{n_0}, p) \leqslant n_0\,\mathrm{L}(U,p).$$

Since the point $p \in X$ is almost-minimizing the ℓ^∞-energy $\mathrm{L}(U)$, we have $\mathrm{L}(U,p) \leqslant \mathrm{L}(U)+\delta$. Since the action of G on X is (κ, N)-acylindrical, it follows from Lemma 2.1.21 and Lemma 2.1.16 that

$$\Phi(G,X) \leqslant N, \quad \text{and} \quad \Delta(g) \leqslant \kappa + (N+2)\,\|g\|^\infty + 100\delta.$$

Using the hypothesis $\mathrm{L}(U) > 10^4 \max\{\kappa, \delta, \alpha\}$ and Equation 2.3.1, we obtain,

$$\max\left\{\frac{\mathrm{L}(U)}{\|g\|^\infty}, \frac{\kappa}{\|g\|^\infty}, \frac{\delta}{\|g\|^\infty}, \frac{\alpha}{\|g\|^\infty}\right\} \leqslant 2.$$

Consequently, we obtain $a_0 \leqslant a_1$ and $b_0 \leqslant b_1$. Let $S = \{\, ug^{b_1}u^{-1} : u \in U^{n_0}(g)\,\}$.

The points (i) and (iii) follow from Proposition 2.3.4 (i) and (iii).

We are going to prove (ii). According to Proposition 2.3.4 (ii), we have $|S| = |U^{n_0}(g)|$. If $|U^{n_0}(g)| = 1$, then $u \sim_g g$, for every $u \in U^{n_0}$. Hence U^{n_0} is contained in $E(g)$. Since U contains the identity, $U \subset U^{n_0}$. Thus H is virtually cyclic (Lemma 2.1.17). Contradiction. Hence $|U^{n_0}(g)| \geqslant 2$. Further, it follows from Proposition 2.3.1 that $|U^{n_0}(g)| \geqslant \frac{1}{a_1}|U^{n_0}|$. Since U contains the identity, $|U^{n_0}| \geqslant |U|$. Therefore,

$$|S| \geqslant \max\left\{2, \frac{1}{a_1}|U|\right\}.$$

This implies our point (ii).

Let's verify the last conclusion about $\omega(U)$. Let $n \geqslant 1$. We have

$$|U^{n_0(b_1+2)n}| \geqslant |S^n| \geqslant |S|^n \geqslant \max\left\{2^n, \left(\frac{1}{a_1}|U|\right)^n\right\},$$

where the first inequality follows from (i); the second from (iii), which implies that the natural homomorphism $\mathbf{F}(S) \to G$ is injective (Proposition 2.4.16); and the third from (ii). Consequently,

$$\omega(U) = \limsup_{n\to\infty} \frac{1}{n_0(b_1+2)n} \log |U^{n_0(b_1+2)n}| \geqslant \frac{1}{n_0(b_1+2)} \max\left\{\log 2, \log\left(\frac{1}{a_1}|U|\right)\right\}.$$

Finally, note that

$$\frac{1}{a_1}|U| \geqslant |U|^{\frac{1}{2}} \Leftrightarrow \log |U| \geqslant 2\log a_1.$$

If $\log |U| \geqslant 2\log a_1$, we obtain

$$\omega(U) \geqslant \frac{1}{n_0(b_1+2)} \log\left(\frac{1}{a_1}|U|\right) \geqslant \frac{1}{2n_0(b_1+2)} \log |U|.$$

If $\log |U| < 2\log a_1$, we obtain

$$\omega(U) \geqslant \frac{1}{n_0(b_1+2)} \log 2 \geqslant \frac{\log 2}{2n_0(b_1+2)\log a_1} \log |U|.$$

□

2.4 Shortening and shortening-free words

In the context of classical small cancellation theory, *Greendlinger's Lemma* states that if a word over the free generating set of a free group represents the identity element in a small cancellation quotient, then it should contain a subword corresponding to a large portion of a relator. This section is structured as follows. First, we are going to formalise the notion of "large portion of a relator" with the definition of shortening word in the context of actions by isometries on hyperbolic spaces. Then, we are going to find a lower bound for the number of shortening-free words of free subgroups generated by reduced subsets of low energy. Finally, we will see that these shortening-free words embedd in geometric small cancellation quotients of appropriate parameters after using a suitable version of Greendlinger's Lemma (Lemma 2.1.33).

Global parameters and hypothesis for this section. Let δ_0 and Δ_0 be the constants of the Small Cancellation Theorem (Lemma 2.1.27). We fix once for all during this section

$$L_0 > 0, \quad \text{and} \quad \tau_0 = 10^6(\delta_0 + L_0 + \Delta_0).$$

Let

$$0 < \delta \leqslant \delta_0, \quad \alpha \geqslant 200\delta_0, \quad \text{and} \quad \tau \geqslant \tau_0.$$

Let G be a group acting by isometries on a δ-hyperbolic space X. Let $U \subset G$ be an α-reduced subset at $p \in X$ (Definition 2.2.1). Let $\mathscr{Q}$ be a loxodromic moving family (Definition 2.1.24). We assume that

$$0 < L(U,p) \leqslant L_0, \quad \text{and} \quad \Delta(\mathscr{Q}, X) \leqslant \Delta_0.$$

2.4.1 Shortening words

Here we study shortening words. Part of this subsection is based on [36, Section 3.1].

Definition 2.4.1 (*Shortening word*). — Let $w \equiv u_1 \cdots u_n$ be an element of $\mathbf{F}(U)$. Let $(H, Y) \in \mathscr{Q}$. We say that w is a *τ-shortening word over* (H, Y) if it satisfies the following. Consider the points $x_0 = p$ and $x_n = wp$. Let y_0 and y_n be respective projections of x_0 and x_n on Y. Then,

(S1) $|y_0 - y_n| > \tau$.

(S2) $|x_0 - y_0| < \frac{1}{2}|u_1 p - p| - 100\delta$, and $|x_n - y_n| < \frac{1}{2}|u_n p - p| - 100\delta$.

A *minimal τ-shortening word over* (H, Y) is a τ-shortening word over (H, Y) none of whose proper prefixes are τ-shortening words over (H, Y).

Remark 2.4.2. — Applying the triangle inequality, we observe that the choice $\tau \geqslant \tau_0$ implies that τ-shortening words over (H, Y) are distinct form the identity:

$$|x_0 - x_n| \geqslant |y_0 - y_n| - |x_0 - y_0| - |x_n - y_n| > 0.$$

Proposition 2.4.3. — *Let $w \equiv u_1 \cdots u_n$ be a τ-shortening word over $(H, Y) \in \mathcal{Q}$. Consider the sequence of $n + 1$ points*

$$x_0 = p, \quad x_1 = u_1 p, \quad x_2 = u_1 u_2 p, \quad \cdots, \quad x_n = u_1 \cdots u_n p.$$

Let y_i be a projection of x_i on Y, for every $i \in [\![0, n]\!]$. Then,

$$|x_i - y_i| < \frac{1}{2} \min\{|u_i p - p|, |u_{i+1} p - p|\} - 100\delta,$$

for every $i \in [\![1, n-1]\!]$.

Proof. — Let $i \in [\![1, n-1]\!]$. Let z_i be a projection of x_i on $[y_0, y_n]$. Since Y is 10δ-quasi-convex (Lemma 2.1.14), there exist $z'_i \in Y$ such that $|z_i - z'_i| \leqslant 11\delta$. By definition,

$$|x_i - y_i| \leqslant d(x_i, Y) + \delta \leqslant |x_i - z'_i| + \delta.$$

By the triangle inequality,

$$|x_i - z'_i| \leqslant |x_i - z_i| + |z_i - z'_i|.$$

By definition, $|x_i - z_i| \leqslant d(x_i, [y_0, y_n]) + \delta$. According to Lemma 2.1.3,

$$d(x_i, [y_0, y_n]) \leqslant (y_0, y_n)_{x_i} + 4\delta.$$

We claim that $(y_0, y_n)_{x_i} \leqslant (x_0, x_n)_{x_i} + 2\delta$. It follows from the four point inequality that

$$\min\{(x_0, y_0)_{x_i}, (y_0, y_n)_{x_i}, (y_n, x_n)_{x_i}\} \leqslant (x_0, x_n)_{x_i} + 2\delta.$$

One can argue using the Broken Geodesic Lemma (Lemma 2.2.3) and the fact that w is a τ-shortening to prove that the minimum must be attained by $(y_0, y_n)_{x_i}$. Now applying the

Broken Geodesic Lemma (Lemma 2.2.3 (ii)),

$$(x_0, x_n)_{x_i} \leqslant (x_{i-1}, x_{i+1})_{x_i} + 2\delta.$$

Moreover, $(x_{i-1}, x_{i+1})_{x_i} = (u_i^{-1}p, u_{i+1}p)_p$. Since the subset U is α-reduced and $\alpha \geqslant 200\delta$,

$$(u_i^{-1}p, u_{i+1}p)_p < \frac{1}{2}\min\{|u_ip - p|, |u_{i+1}p - p|\} - 118\delta.$$

Combining all the estimations, we obtain

$$|x_i - y_i| < \frac{1}{2}\min\{|u_ip - p|, |u_{i+1}p - p|\} - 100\delta.$$

□

Proposition 2.4.4. — *Let $w \equiv u_1 \cdots u_n$ be a τ-shortening word over $(H, Y) \in \mathcal{Q}$. The following holds.*

(i) We have

$$|w|_U \geqslant \frac{\tau - 50\delta}{\mathrm{L}(U, p)}.$$

(ii) If w is a minimal τ-shortening word over (H, Y), then

$$|w|_U \leqslant \frac{\tau}{\alpha} + 2.$$

Proof. — Consider the sequence of $n + 1$ points

$$x_0 = p, \quad x_1 = u_1p, \quad x_2 = u_1u_2p, \quad \cdots, \quad x_n = u_1 \cdots u_np.$$

Let y_i be a projection of x_i on Y, for every $i \in [\![0, n]\!]$.

(i) Since $\mathrm{L}(U, p) > 0$ and w is distinct from the identity (Remark 2.4.2), it follows from the triangle inequality that,

$$|w|_U \geqslant \frac{|x_0 - x_n|}{\mathrm{L}(U, p)}.$$

According to (S1), we have $|y_0 - y_n| > \tau$. Since Y is 10δ-quasi-convex (Lemma 2.1.14) and $\tau \geqslant 23\delta$, the strong contraction property of Y (Lemma 2.1.6) implies

$$|x_0 - x_n| \geqslant |x_0 - y_0| + |y_0 - y_n| + |y_n - x_n| - 46\delta.$$

Consequently, $|x_0 - x_n| > \tau - 50\delta$. Therefore,

$$|w|_U \geqslant \frac{\tau - 50\delta}{\mathrm{L}(U,p)}.$$

(ii) Assume that w is a minimal τ-shortening word over (H, Y). Let $w' \equiv u_1 \cdots u_{n-1}$. By definition,

$$|w|_U = |w'|_U + 1.$$

In view of Proposition 2.2.4 (ii), we deduce

$$|w'p - p| \geqslant \frac{1}{2}|u_1 p - p| + \frac{1}{2}|u_{n-1}p - p| + \alpha(|w'|_U - 1).$$

By the triangle inequality,

$$|w'p - p| \leqslant |x_{n-1} - y_{n-1}| + |y_{n-1} - y_0| + |y_0 - x_0|.$$

Since w is a τ-shortening word over (H, Y), the property (S2) implies

$$|x_0 - y_0| < \frac{1}{2}|u_1 p - p| - 100\delta.$$

According to Proposition 2.4.3,

$$|x_{n-1} - y_{n-1}| < \frac{1}{2}|u_{n-1}p - p| - 100\delta.$$

Therefore, since w' is not a τ-shortening over (H, Y), we have $|y_{n-1} - y_0| \leqslant \tau$. Consequently, $|w'|_U \leqslant \frac{\tau}{\alpha} + 1$. Thus, $|w|_U \leqslant \frac{\tau}{\alpha} + 2$.

□

Proposition 2.4.5. — *Let $(H_1, Y_1), (H_2, Y_2) \in \mathscr{Q}$. Let $w \in \mathbf{F}(U)$. If w is a τ-shortening word over both (H_1, Y_1) and (H_2, Y_2), then $(H_1, Y_1) = (H_2, Y_2)$.*

Proof. — Assume that w is a τ-shortening word over (H_1, Y_1) and (H_2, Y_2). In order to prove that $(H_1, Y_1) = (H_2, Y_2)$, it is enough to show that $\mathrm{diam}(Y_1^{+20\delta} \cap Y_2^{+20\delta}) > \Delta(\mathscr{Q}, X)$. Since the subsets Y_1 and Y_2 are 10δ-quasi-convex (Lemma 2.1.14), it follows from Lemma 2.1.8 that

$$\mathrm{diam}(Y_1^{+20\delta} \cap Y_2^{+20\delta}) \geqslant \mathrm{diam}(Y_1^{+13\delta} \cap Y_2^{+13\delta}) \geqslant \mathrm{diam}(Y_1^{+2L_0} \cap Y_2^{+2L_0}) - 4L_0 - 4\delta_0.$$

Let $i \in [\![1, 2]\!]$. Let x_i and z_i be respective projections of p and wp on Y_i. We claim that $x_1, z_1 \in Y_1^{+2L_0} \cap Y_2^{+2L_0}$. Since w is a shortening word over (H_i, Y_i), it follows from (S2) that

$$\max\{|p - x_i|, |wp - z_i|\} \leqslant L_0.$$

According to the triangle inequality,

$$|x_1 - x_2| \leqslant |x_1 - p| + |p - x_2|, \quad |z_1 - z_2| \leqslant |z_1 - p| + |p - z_2|.$$

Consequently,

$$\max\{|x_1 - x_2|, |z_1 - z_2|\} \leqslant 2L_0.$$

Therefore, $x_1, z_1 \in Y_2^{+2L_0}$. This proves the claim. Thus,

$$\operatorname{diam}(Y_1^{+2L_0} \cap Y_2^{+2L_0}) \geqslant |x_1 - z_1|.$$

Since w is a shortening over (H_1, Y_1), it follows from (S1) that $|x_1 - z_1| > \tau$. Finally, since $\tau \geqslant \tau_0$, we obtain that $\operatorname{diam}(Y_1^{+20\delta} \cap Y_2^{+20\delta}) > \Delta(\mathcal{Q}, X)$. □

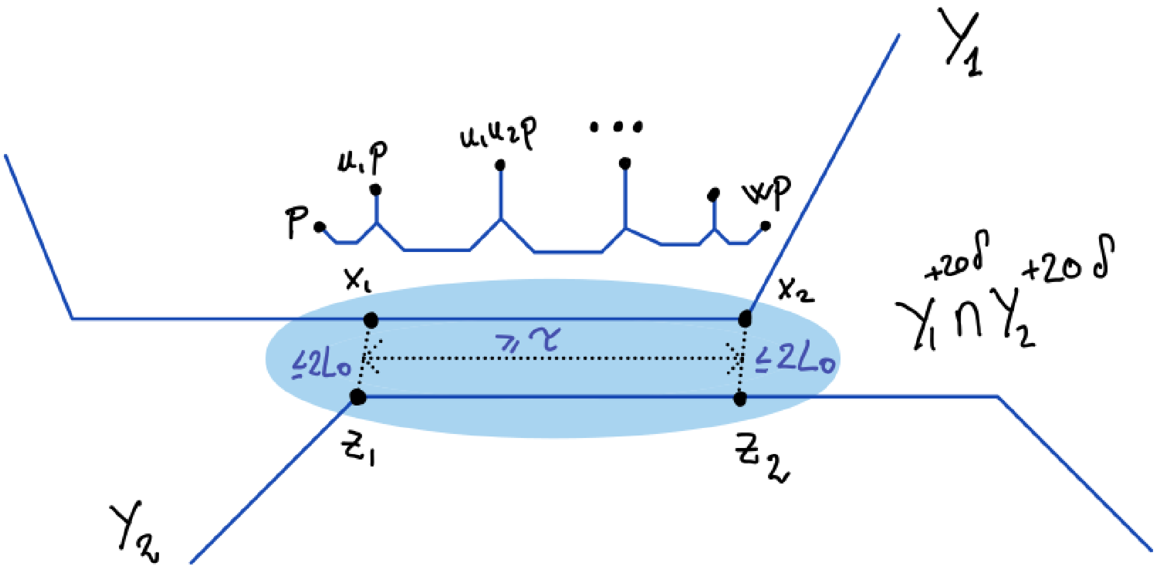

Figure 2.3 – Scheme for the proof of Proposition 2.4.5.

Proposition 2.4.6. — *For every $(H, Y) \in \mathcal{Q}$, there exist at most two minimal τ-shortening words over (H, Y).*

Proof. — Let $(H, Y) \in \mathcal{Q}$. Let η^- and η^+ be the points of ∂X fixed by H and $\gamma\colon \mathbf{R} \to X$ be an $10^3\delta$-local $(1, \delta)$-quasi-geodesic joining η^- to η^+. Let q be a projection of p on γ. Without loss of generality, we may assume that $q = \gamma(0)$. Let $\mathscr{S}_{(H,Y)}$ denote the set of elements in $\mathbf{F}(U)$ that are τ-shortening words over (H, Y). Assume that $\mathscr{S}_{(H,Y)}$ is non-empty, otherwise the statement is true. We decompose $\mathscr{S}_{(H,Y)}$ in two sets as follows: an element $w \in \mathscr{S}_{(H,Y)}$ belongs to $\mathscr{S}^+_{(H,Y)}$ (respectively, $\mathscr{S}^-_{(H,Y)}$) if there is a projection $\gamma(t)$ of wp on γ with $t \geqslant 0$ (respectively, $t \leqslant 0$). Observe that a priori the sets $\mathscr{S}^-_{(H,Y)}$ and $\mathscr{S}^+_{(H,Y)}$ are not disjoint, but that will not be an issue for the rest of the proof.

Let $w_1, w_2 \in \mathscr{S}^+_{(H,Y)}$. Let $q_1 = \gamma(t_1)$ and $q_2 = \gamma(t_2)$ be the respective projections of $w_1 p$ and $w_2 p$ on γ. Without loss of generality, we may assume that $0 \leqslant t_1 \leqslant t_2$.

Claim 2.4.7. — The word w_1 is a prefix of w_2.

Proof. — We are going to apply the Geodesic Extension Property (Proposition 2.2.6). By the triangle inequality,

$$(p, w_2 p)_{w_1 p} \leqslant |w_1 p - q_1| + (w_2 p, p)_{q_1}. \tag{2.4.1}$$

Assume that $w_1 \equiv u_1 \cdots u_m$.

(a) Let's estimate $|w_1 p - q_1|$. By definition, the H-invariant cylinder Y is contained in the 20δ-neighbourhood of γ. Consequently,

$$|w_1 p - q_1| = d(w_1 p, \gamma) \leqslant d(w_1 p, Y) + 20\delta.$$

Since w_1 is a τ-shortening word over (H, Y), the property (S2) implies

$$d(w_1 p, Y) < \frac{1}{2}|u_m p - p| - 100\delta.$$

Therefore,

$$|w_1 p - q_1| < \frac{1}{2}|u_m p - p| - 80\delta. \tag{2.4.2}$$

(b) Let's estimate $(w_2 p, p)_{q_1}$. By definition,

$$(w_2 p, p)_{q_1} = \frac{1}{2}(|w_2 p - q_1| + |p - q_1| - |w_2 p - p|).$$

Since w_2 is a τ-shortening word over (H, Y), the property (S1) implies

$$|q_2 - q| > \tau.$$

Since Y is 10δ-quasi-convex (Lemma 2.1.14) and $\tau \geqslant 23\delta$, the strong contraction property of Y (Lemma 2.1.6) implies

$$|w_2 p - p| \geqslant |w_2 p - q_2| + |q_2 - q| + |q - p| - 46\delta.$$

Again by definition,

$$|q_2 - q| = |q_2 - q_1| + |q_1 - q| - 2(q_2, q)_{q_1}.$$

According to Lemma 2.1.15 (i),

$$(q_2, q)_{q_1} \leqslant 6\delta.$$

Note that here we have used the assumption $0 \leqslant t_1 \leqslant t_2$. By the triangle inequality,

$$|w_2 p - q_1| \leqslant |w_2 p - q_2| + |q_2 - q_1|.$$

Therefore,

$$|w_2 p - p| \geqslant |w_2 p - q_1| + |q_1 - p| - 58\delta.$$

Consequently,

$$(w_2 p, p)_{q_1} \leqslant 29\delta. \tag{2.4.3}$$

Finally, combining Equation 2.4.1, Equation 2.4.2 and Equation 2.4.3, we obtain

$$(p, w_2 p)_{w_1 p} \leqslant \frac{1}{2}|u_m p - p| - \delta.$$

Therefore, the Geodesic Extension Property (Proposition 2.2.6) implies that w_1 is a prefix of w_2. This proves our claim. □

If w_1 is not a proper prefix of w_2, then the claim above implies that $w_1 = w_2$. Therefore $\mathscr{S}^+_{(H,Y)}$ has at most one element satisfying the statement of the proposition. By symmetry, $\mathscr{S}^-_{(H,Y)}$ has at most one element satisfying the statement. Therefore $\mathscr{S}_{(H,Y)}$ has at most two elements satisfying the statement.

□

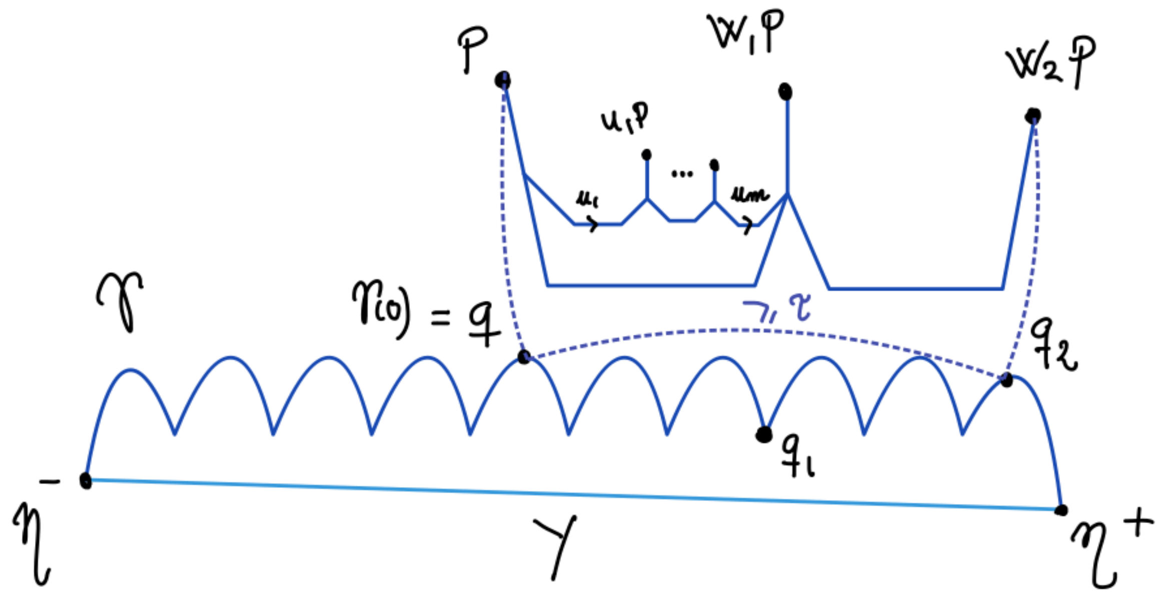

Figure 2.4 – Scheme for the proof of Proposition 2.4.6.

2.4.2 The growth of shortening-free words

Here we count shortening-free words. The counting is based on [36, Section 3.22].

Definition 2.4.8 (*Shortening-free word*). — Let $w \equiv u_1 \cdots u_n$ be an element of $\mathbf{F}(U)$. Let $(H, Y) \in \mathscr{Q}$. We say that w *contains a τ-shortening word over* (H, Y) if w splits as $w \equiv w_0 w_1 w_2$, where w_1 is a τ-shortening word over (H, Y). We say that w is a *τ-shortening-free word* if for every $(H, Y) \in \mathscr{Q}$, the word w does not contain any τ-shortening word over (H, Y). We denote by $F(\tau) \subset \mathbf{F}(U)$ the subset of τ-shortening-free words.

Recall that the natural homomorphism $\mathbf{F}(U) \to G$ is injective (Proposition 2.2.5). Hence, we can safely identify the elements of $\mathbf{F}(U)$ with their images in G. The *ball* $B_U(n) \subset \mathbf{F}(U)$ of radius n is the set of reduced words over the alphabet $U \sqcup U^{-1}$ of length $|w|_U \leqslant n$, for every $n \geqslant 0$. Note that $B_U(n) = (U \sqcup U^{-1} \sqcup \{1\})^n$ when $n \geqslant 1$. Recall that we have fixed global hypothesis at the beginning of this section. The goal of this subsection is to obtain the following estimation.

Proposition 2.4.9. — *For every $\theta \in (0, 1/2)$, there exist $\tau_1 \geqslant \tau_0$ depending on θ, δ_0, L_0 and Δ_0 with the following property. If $|U| \geqslant 2$ and $\tau \geqslant \tau_1$, then for every $n \geqslant 0$, we have*

$$|F(\tau) \cap B_U(n + 1)| \geqslant (1 - \theta)(2|U| - 1)|F(\tau) \cap B_U(n)|.$$

In particular, for every $n \geqslant 0$

$$|F(\tau) \cap B_U(n)| \geqslant (1-\theta)^n (2|U|-1)^n.$$

We are going to divide the proof of Proposition 2.4.9 into a few lemmas. First we fix some notations. We let

$$Z = \{\, w \in \mathbf{F}(U) \,:\, w \equiv w_0 u, w_0 \in F(\tau), u \in U \sqcup U^{-1} \,\}.$$

For every $(H, Y) \in \mathscr{Q}$, we denote by $Z_{(H,Y)} \subset Z$ the set of elements $w \in Z$ that split as $w \equiv w_1 w_2$, where $w_1 \in F(\tau)$ and w_2 is a τ-shortening word over (H, Y).

Lemma 2.4.10. — *The set Z is contained in the disjoint union of $F(\tau)$ and $\bigcup_{(H,Y)\in\mathscr{Q}} Z_{(H,Y)}$.*

Proof. — The sets $F(\tau)$ and $\bigcup_{(H,Y)\in\mathscr{Q}} Z_{(H,Y)}$ are disjoint as a direct consequence of the definitions. Let $w \in Z - F(\tau)$. Since $w \in Z$, there exist $w_0 \in F(\tau)$ and $u \in U \sqcup U^{-1}$ such that $w \equiv w_0 u$. Since $w \notin F(\tau)$, there exist $(H, Y) \in \mathscr{Q}$ and a subword w_2 of w that is a τ-shortening word over (H, Y). It follows from the definition of $F(\tau)$ that every subword of w_0 must also be in $F(\tau)$. In particular, the word w_2 cannot be a subword of w_0. Hence, the only possibility is that w_2 is a suffix of w. Therefore, $w \in Z_{(H,Y)}$. □

Our Lemma 2.4.10 implies that for every $n \geqslant 0$,

$$|F(\tau) \cap B_U(n)| \geqslant |Z \cap B_U(n)| - \sum_{(H,Y)\in\mathscr{Q}} |Z_{(H,Y)} \cap B_U(n)|. \tag{2.4.4}$$

The next step is to estimate each term in the right side of the above inequality. The following lemma is a direct consequence of the definition of Z.

Lemma 2.4.11. — *For every $n \geqslant 0$,*

$$|Z \cap B_U(n+1)| = (2|U|-1)|F(\tau) \cap B_U(n)|.$$

Lemma 2.4.12. — *Let*

$$a = 2, \quad b = \left\lceil \frac{\tau_0}{200\delta_0} + 2 \right\rceil + 1, \quad M = \left\lfloor \frac{\tau - 50\delta_0}{L_0} \right\rfloor.$$

If $|U| \geqslant 2$, then for every $n \geqslant 0$,

$$\sum_{(H,Y)\in\mathscr{Q}} |Z_{(H,Y)} \cap B_U(n)| \leqslant a(2|U|-1)^b |F(\tau) \cap B_U(n-M)|.$$

Proof. — Assume that $|U| \geqslant 2$. Let $n \geqslant 0$. Note that for every $(H,Y) \in \mathscr{Q}$, the set $Z_{(H,Y)}$ is empty whenever there is no τ-shortening word over (H,Y). We denote by $\mathscr{Q}_0$ the set of $(H,Y) \in \mathscr{Q}$ for which there exist a τ-shortening word over (H,Y). We have,

$$\sum_{H\in\mathscr{Q}} |Z_{(H,Y)} \cap B_U(n)| = \sum_{(H,Y)\in\mathscr{Q}_0} |Z_{(H,Y)} \cap B_U(n)|.$$

The desired estimation is obtained from the two estimations of the claims below:

Claim 2.4.13. — $|Z_{(H,Y)} \cap B_U(n)| \leqslant a|F(\tau) \cap B_U(n-M)|$, for every $(H,Y) \in \mathscr{Q}_0$.

Proof. — Let $(H,Y) \in \mathscr{Q}_0$. Let $w \in Z_{(H,Y)} \cap B_U(n)$. Since $w \in Z_{(H,Y)}$, there exist $w_1 \in F(\tau)$ and a τ-shortening word w_2 over (H,Y) such that $w \equiv w_1 w_2$. We are going to describe the possible choices of w_1 and w_2. Since w is a reduced word over $U \sqcup U^{-1}$,

$$|w_1|_U = |w|_U - |w_2|_U.$$

According to Proposition 2.4.4 (i),

$$|w_2|_U \geqslant \frac{\tau - 50\delta_0}{L_0} \geqslant M \geqslant 0.$$

Therefore, $w_1 \in F(\tau) \cap B_U(n-M)$. Since $w \in Z$, the prefix consisting of all but the last letter is a τ-shortening free word. Thus, no proper prefix of w_2 is a τ-shortening word. It follows from Proposition 2.4.6 that there are most $a = 2$ possible choices for w_2. Therefore, there are at most $a|F(\tau) \cap B_U(n-M)|$ choices for w. This proves our claim. □

Claim 2.4.14. — $|\mathscr{Q}_0| \leqslant (2|U|-1)^b$

Proof. — Let $d = \left\lceil \frac{\tau_0}{200\delta_0} + 2 \right\rceil$. Since the free group $\mathbf{F}(U)$ has rank $|U| \geqslant 2$, we have

$$|B_U(d)| = \frac{|U|(2|U|-1)^d - 1}{|U|-1} \leqslant (2|U|-1)^{d+1} = (2|U|-1)^b.$$

Consequently, it suffices to show that there exists an injective map $\chi\colon \mathscr{Q}_0 \to B_U(d)$. Let $(H,Y) \in \mathscr{Q}_0$. By definition, there exist a τ-shortening word w over (H,Y). Note that since

$\tau \geqslant \tau_0$, we have that w is a τ_0-shortening word over (H, Y). Let w' be the shortest prefix of w that is a τ_0-shortening word over (H, Y). In particular, w' is a minimal τ_0-shortening word over (H, Y). We define $\chi(H, Y) = w'$. Since $\alpha \geqslant 200\delta_0$, according to Proposition 2.4.4 (ii), $|w'|_U \leqslant d$. According to Proposition 2.4.5, there exist at most one $(H, Y) \in \mathscr{Q}$ such that w' is a τ_0-shortening word over (H, Y). Hence χ is well-defined and injective. This proves our claim. □

□

Lemma 2.4.15. — *For every $\theta \in (0, 1/2)$ and a, $b \geqslant 1$, there exist $M_0 \geqslant 0$ with the following property. Let*

$$\mu = (1-\theta)(2|U|-1), \quad \xi = a(2|U|-1)^b, \quad \text{and } \sigma = \frac{\theta}{2(1-\theta)\xi}.$$

If $|U| \geqslant 2$, then for every $M \geqslant M_0$, we have

$$\frac{1}{\mu^M} \leqslant \sigma.$$

Proof. — Let $\theta \in (0, 1/2)$ and a, $b \geqslant 1$. Let $M_0 = \max\left\{b, \frac{d_1}{d_2}\right\}$, where d_1, d_2 are constants depending only on θ, a, b whose exact value will be precised below. Let μ, ξ, σ as above. Assume that $|U| \geqslant 2$. Let $M \geqslant M_0$. In order to prove that $\frac{1}{\mu^M} \leqslant \sigma$, it is enough to show that $\log\left(\frac{1}{\sigma\mu^M}\right) \leqslant 0$. A first computation yields

$$\begin{aligned} \log\left(\frac{1}{\sigma\mu^M}\right) &= -\log\sigma - \log\left(\mu^M\right), \\ \log(\sigma) &= \log\left(\frac{\theta}{2(1-\theta)a}\right) - b\log(2|U|-1), \\ \log\left(\mu^M\right) &= M\log(1-\theta) + M\log(2|U|-1). \end{aligned}$$

Consequently,

$$\log\left(\frac{1}{\sigma\mu^M}\right) \leqslant (b-M)\log(2|U|-1) - M\log(1-\theta) - \log\left(\frac{\theta}{2(1-\theta)a}\right).$$

Since $M \geqslant b$ and $|U| \geqslant 2$, we have

$$(b-M)\log(2|U|-1) \leqslant (b-M)\log 3.$$

Therefore,

$$\log\left(\frac{1}{\sigma\mu^M}\right) \leqslant -M[\log 3 + \log(1-\theta)] + b\log 3 - \log\left(\frac{\theta}{2(1-\theta)a}\right).$$

We put

$$d_1 = b\log 3 + \log(2a) - \log\left(\frac{\theta}{1-\theta}\right), \quad d_2 = \log 3 + \log(1-\theta).$$

Since $a \geqslant 1$, $b \geqslant 1$ and $\theta \in (0, 1/2)$, we have $\min\{d_1, d_2\} > 0$. Finally, since $M \geqslant \frac{d_1}{d_2}$, we obtain, $\log\left(\frac{1}{\sigma\mu^M}\right) \leqslant 0$. □

We are ready to prove the proposition.

Proof of Proposition 2.4.9. — Let $\theta \in (0, 1/2)$. We are going to define the constant τ_1. Let

$$a = 2, \quad b = \left\lceil \frac{\tau_0}{200\delta_0} + 2 \right\rceil + 1.$$

Let $M_0 \geqslant 0$ be the constant of Lemma 2.4.15 depending on θ, a, b. We put

$$\tau_1 = \max\{\tau_0, L_0(M_0 + 1) + 50\delta_0\}.$$

Assume that $|U| \geqslant 2$ and $\tau \geqslant \tau_1$. We define the auxiliary parameters

$$\mu = (1-\theta)(2|U|-1), \quad \xi = a(2|U|-1)^b, \quad \sigma = \frac{\theta}{2\xi(1-\theta)}, \quad \text{and } M = \left\lfloor \frac{\tau - 50\delta_0}{L_0} \right\rfloor.$$

In particular, $M \geqslant M_0$. For every $n \geqslant 0$, we let

$$c(n) = |F(\tau) \cap B_U(n)|.$$

We must prove that for every $n \geqslant 1$,

$$c(n) \geqslant \mu c(n-1).$$

The proof goes by induction on n:

Base step. We claim that $c(1) \geqslant \mu$. Note that $B_U(1) = U \sqcup U^{-1} \sqcup \{1\}$. Therefore, it is enough to show that $U \sqcup U^{-1} \sqcup \{1\}$ is contained in $F(\tau)$. Let $w \in U \sqcup U^{-1} \sqcup \{1\}$. In particular, $|w|_U = 1$. Therefore, $w \in F(\tau)$ if and only if for every $(H, Y) \in \mathscr{Q}$, the element w is not a τ-shortening word over (H, Y). According to Proposition 2.4.4 (i), for every

$(H, Y) \in \mathcal{Q}$ and for every τ-shortening word v over (H, Y), we have $|v|_U \geqslant \frac{\tau - 50\delta_0}{L_0}$. Since $\tau \geqslant \tau_0$, we have $1 < \frac{\tau - 50\delta_0}{L_0}$. Consequently, $w \in F(\tau)$. This proves our claim.

Inductive step. Let $n \geqslant 1$. Assume that $c(m) \geqslant \mu c(m-1)$, for every $m \in [\![1, n]\!]$. We claim that $c(n+1) \geqslant \mu c(n)$. According to Equation 2.4.4,

$$c(n+1) \geqslant |Z \cap B_U(n+1)| - \sum_{(H,Y) \in \mathcal{Q}} |Z_{(H,Y)} \cap B_U(n+1)|.$$

It follows from Lemma 2.4.11 and Lemma 2.4.12 that

$$c(n+1) \geqslant (2|U| - 1)c(n) - \xi c(n+1-M).$$

The induction hypothesis implies that for every $k \geqslant 0$, we have $c(n-k) \leqslant \mu^{-k} c(n)$. Note that $M - 1 \geqslant 0$. Therefore, specifying the choice $k = M - 1$, we obtain

$$c(n+1) \geqslant \left(1 - \frac{\xi\mu}{2|U| - 1} \frac{1}{\mu^M}\right)(2|U| - 1)c(n).$$

Recall that we defined $\mu = (1 - \theta)(2|U| - 1)$. Hence, in order to prove our claim, it is enough to show that

$$\frac{\xi\mu}{2|U| - 1} \frac{1}{\mu^M} \leqslant \theta.$$

Since $M \geqslant M_0$, it follows from Lemma 2.4.15 that

$$\frac{1}{\mu^M} \leqslant \sigma.$$

Finally, note that

$$\frac{\xi\mu}{2|U| - 1}\sigma = \frac{\xi(1-\theta)(2|U| - 1)}{2|U| - 1} \frac{\theta}{2\xi(1-\theta)} = \frac{\theta}{2} \leqslant \theta.$$

This proves our claim. □

2.4.3 The injection of shortening-free words

Let ρ_0 be the constant of the Small Cancellation Theorem (Lemma 2.1.27). Let $\tau_1 \geqslant \tau_0$ be the constant of Proposition 2.4.9 depending on $\theta = 1/3$, δ_0, L_0 and Δ_0. Let

$$\rho \geqslant \max\{\rho_0, \log(2[4\tau_1 + 23\delta_0] + 1)\}.$$

In addition to the global hypothesis for this section, we assume that

$$\mathrm{T}(\mathscr{Q}, X) \geqslant 100\pi \sinh \rho.$$

Denote $K = \langle\langle H \mid (H,Y) \in \mathscr{Q} \rangle\rangle$ and $\bar{G} = G/K$. The goal of this subsection is to prove:

Proposition 2.4.16. — *There exists $\tau_2 \geqslant \tau_1$ depending on δ_0, L_0 and Δ_0 with the following property. The restriction of the natural homomorphism $\mathbf{F}(U) \to \overline{G}$ to the subset of τ_2-shortening-free words is an injection.*

Lemma 2.4.17. — *Let $w \equiv u_1 \cdots u_n$ be an element of $\mathbf{F}(U)$. Let $(H,Y) \in \mathscr{Q}$. Let y_0 and y_n be respective projections of p and wp on Y. If $|y_0 - y_n| > 2\tau$, then w contains a $(2\tau - \tau_0)$-shortening word over a conjugate of (H,Y).*

Proof. — Consider the sequence of $n+1$ points

$$x_0 = p, \quad x_1 = u_1 p, \quad x_2 = u_1 u_2 p, \quad \cdots, \quad x_n = u_1 \cdots u_n p.$$

Let y_i be a projection of x_i on Y, for every $i \in [\![0, n]\!]$. Assume that $|y_0 - y_n| > 2\tau$. Since Y is 10δ-quasi-convex (Lemma 2.1.14) and $\tau \geqslant 23\delta$, the strong contraction property of Y (Lemma 2.1.6) implies that there exist $y_0', y_n' \in [p, wp]$ such that

$$\max\{|y_0 - y_0'|, |y_n - y_n'|\} \leqslant 23\delta \leqslant 23\delta_0.$$

Consider the broken geodesic

$$\gamma_w = \bigcup_{i=1}^{n} (u_1 \cdots u_{i-1})[p, u_i p].$$

Let y_0'' and y_n'' be respective projections of y_0' and y_n' on γ_w. Up to permuting y_0' and y_n' we may assume that p, y_0'', y_n'' and wp are ordered in this way along γ_w. In particular, there are $i \leqslant n-1$ and $j \leqslant n-1$ such that $y_0'' \in (u_1 \cdots u_i)[p, u_{i+1}p]$ and $y_n'' \in (u_1 \cdots u_j)[p, u_{j+1}p]$. Since y_0'' comes before y_n'' on γ_w, we have $i \leqslant j$. Let $w_0 \equiv u_1 \cdots u_{i+1}$ and take the word w_1 such that $w_0 w_1 \equiv u_1 \cdots u_j$. We are going to prove that w_1 is a $(2\tau - \tau_0)$-shortening word over $(w_0^{-1} H w_0, w_0^{-1} Y)$. The property (S2) follows from the fact that U is $200\delta_0$-reduced at p and from the Broken Geodesic Lemma (Lemma 2.2.3). Let's prove (S1), i.e. $|y_{i+1} - y_j| > 2\tau - \tau_0$. By the triangle inequality,

$$|y_{i+1} - y_j| \geqslant |y_0 - y_n| - |y_0 - y_{i+1}| - |y_n - y_j|,$$

$$|y_0 - y_{i+1}| \leqslant |y_0 - y_0'| + |y_0' - y_0''| + |y_0'' - x_{i+1}| + |x_{i+1} - y_{i+1}|,$$
$$|y_n - y_j| \leqslant |y_n - y_n'| + |y_n' - y_n''| + |y_n'' - x_j| + |x_j - y_j|.$$

Since $[x_0, x_n]$ is contained in the 5δ-neighbouhood of γ_w (Lemma 2.2.3 (iii)),

$$\max\{|y_0' - y_0''|, |y_n' - y_n''|\} \leqslant 5\delta \leqslant 5\delta_0.$$

Since $y_0'' \in (u_1 \cdots u_i)[p, u_{i+1}p]$ and $y_n'' \in (u_1 \cdots u_j)[p, u_{j+1}p]$,

$$\max\{|y_0'' - x_{i+1}|, |y_n'' - x_j|\} \leqslant \mathrm{L}(U, p) \leqslant L_0.$$

It follows from (S2) that,

$$\max\{|x_{i+1} - y_{i+1}|, |x_j - y_j|\} \leqslant \mathrm{L}(U, p) \leqslant L_0.$$

Combining the previous estimations, we obtain $|y_{i+1} - y_j| > 2\tau - \tau_0$. Note that $2\tau - \tau_0 \geqslant \tau_0$. □

Proof of Proposition 2.4.16. — We put $\tau_2 = 2\tau_1 - \tau_0$. Let $w_1, w_2 \in \mathbf{F}(U)$ be two τ_2-shortening-free words such that $w_1 w_2 \in K$. Assume for a contradiction that $w_1 w_2$ is not the identity as an element of G. According to Greendlinger's Lemma (Lemma 2.1.33), there exist $(H, Y) \in \mathscr{Q}$ such that if y_0 and y_2 are respective projections of p and $w_1 w_2 p$ on Y, then

$$|y_0 - y_2| > T(H, X) - 2\pi \sinh \rho - 23\delta.$$

By definition, $T(H, X) \geqslant T(\mathscr{Q}, X)$. By hypothesis

$$T(\mathscr{Q}, X) \geqslant 100\pi \sinh \rho, \quad \text{and} \quad \delta \leqslant \delta_0.$$

Therefore,

$$|y_0 - y_2| > \frac{e^\rho - 1}{2} - 23\delta_0.$$

The choice of ρ now implies that

$$|y_0 - y_2| > 4\tau_1$$

Let y_1 be a projection of $w_1 p$ on Y. Note that $w_1^{-1} y_1$ and $w_1^{-1} y_2$ are respective projections of p and $w_2 p$ on $w_1^{-1} Y$. Also, $(w_1^{-1} H w_1, w_1^{-1} Y) \in \mathscr{Q}$. Since w_1 and w_2 are τ_2-shortening-free words, it follows from Lemma 2.4.17 that

$$\max\{|y_0 - y_1|, |y_1 - y_2|\} < 2\tau_1.$$

By the triangle inequality,

$$|y_0 - y_2| \leqslant |y_0 - y_1| + |y_1 - y_2| \leqslant 4\tau_1.$$

Contradiction. Hence $w_1 w_2 = 1$. □

2.5 Growth in small cancellation groups

The goal of this section is to prove Theorem 0.6.2. We start with the following lemma.

Lemma 2.5.1. — *Let $a \geqslant 0$, $b \geqslant a$. Let G be a group acting acylindrically on a δ-hyperbolic space X. Let $U \subset G$ be a finite symmetric subset containing the identity such that $\mathrm{L}(U) \leqslant b$. Let $\Gamma = \langle U \rangle$. One of the following holds.*

(i) Γ is elliptic.

(ii) There exist $n \geqslant 1$ depending on U such that

$$a < \mathrm{L}(U^n) \leqslant 2b.$$

Proof. — Assume that Γ is not elliptic. Since the action of G on X is acylindrical, there exists a loxodromic element $g \in \Gamma$ (Lemma 2.1.22).

Claim 2.5.2. — There exists $M_0 \geqslant 1$ depending on U such that for every $M \geqslant M_0$,

$$\mathrm{L}(U^M) > a.$$

Proof. — According to Lemma 2.1.13, the global injectivity radius $\mathrm{T}(G, X)$ is distinct from zero. Let $m \geqslant \frac{a+\delta}{\mathrm{T}(G,X)}$. Since $g \in \Gamma$ and U is a symmetric generating set, there exists $M_0 \geqslant 1$ depending on U such that $g^m \in U^{M_0}$. Let $M \geqslant M_0$. Let $p \in X$ almost-minimizing the ℓ^∞-energy $\mathrm{L}(U^{M_0})$. We have,

$$\mathrm{L}(U^M, p) \geqslant \mathrm{L}(U^{M_0}, p) \geqslant |g^m p - p| \geqslant \|g^m\|^\infty = m\|g\|^\infty > a + \delta.$$

Hence $\mathrm{L}(U^M) > a$. This proves our claim. □

It follows from the claim above that there exist a smallest number $n \geqslant 1$ depending on U such that $\mathrm{L}(U^n) > a$. If $n = 1$, then we have $\mathrm{L}(U) \leqslant b$ by hypothesis. Therefore, $\mathrm{L}(U) \leqslant 2b$. If $n \geqslant 2$, then $n \leqslant 2(n-1)$. Since U contains the identity, $U^n \subset U^{2(n-1)}$. By the triangle inequality,

$$\mathrm{L}(U^n) \leqslant \mathrm{L}(U^{2(n-1)}) \leqslant 2\,\mathrm{L}(U^{n-1}) \leqslant 2a \leqslant 2b.$$

□

Hypothesis for the remainder of this section. Recall that the constants of the Small Cancellation Theorem (Lemma 2.1.27) are δ_0, $\bar{\delta}$, Δ_0, ρ_0. We can choose δ_0 arbitrarily small (Remark 2.1.28). For convenience, we will assume

$$\delta_0 \leqslant \frac{\pi \sinh 10^4 \bar{\delta}}{10^4 \cdot 200}.$$

We define the first geometric small cancellation parameter:

$$\lambda \leqslant \frac{\Delta_0}{100\pi \sinh \rho_0}.$$

Let $N > 0$. Let $c > 1$ be the constant of Theorem 2.3.8 depending only on the acylindricity parameters (δ_0, N). We fix an auxiliar parameter that will be used to bound the ℓ^∞-energy:

$$L_0 = c \cdot (2\pi \sinh 10^4 \bar{\delta} + \delta_0).$$

Let τ_1 and τ_2 be the constants of Proposition 2.4.16 depending on δ_0, L_0 and Δ_0. Let

$$\rho \geqslant \max\left\{\rho_0, \log(2[4\tau_1 + 23\delta_0] + 1), 5 \cdot 10^4 \bar{\delta}\right\}.$$

Let $\delta > 0$ and $\kappa \geqslant \delta$. We define the second geometric small cancellation parameter:

$$\varepsilon \geqslant \frac{100\pi \sinh \rho}{\delta_0} \cdot \frac{\kappa}{\delta}.$$

Let G be a group acting (κ, N)-acylindrically on a δ-hyperbolic space X. Let $\mathscr{Q}$ be a loxodromic moving family satisfying the geometric $C''(\lambda, \varepsilon)$-small cancellation condition for the action of G on X. We define a rescaling parameter

$$\sigma = \min\left\{\frac{\delta_0}{\kappa}, \frac{\Delta_0}{\Delta(\mathcal{Q}, X)}\right\}.$$

Remark 2.5.3. — Instead of working with the action of G on X, we will work with the action of G on the rescaled space $\mathcal{X}$.

The space $\mathcal{X}$ is $\sigma\delta$-hyperbolic and the action of G on $\mathcal{X}$ is $(\sigma\kappa, N)$-acylindrical. Note that

$$\sigma\delta \leqslant \sigma\kappa \leqslant \delta_0,$$

where the first inequality comes from the hypothesis $\kappa \geqslant \delta$. In particular, the action of G on $\mathcal{X}$ is (δ_0, N)-acylindrical for the hyperbolicity constant $\sigma\delta$. Besides, we have

$$\begin{aligned} &\Delta(\mathcal{Q}, \mathcal{X}) \leqslant \sigma\Delta(\mathcal{Q}, X) \leqslant \Delta_0, \\ &\mathrm{T}(\mathcal{Q}, \mathcal{X}) \geqslant \sigma\, \mathrm{T}(\mathcal{Q}, X) \geqslant \sigma \max\left\{\varepsilon\delta, \frac{\Delta(\mathcal{Q}, X)}{\lambda}\right\} \geqslant 100\pi \sinh \rho. \end{aligned}$$

Note that the second equation is deduced after using the geometric $C''(\lambda, \varepsilon)$-small cancellation condition. Therefore G, $\mathcal{X}$ and $\mathcal{Q}$ satisfy the hypothesis of the Small Cancellation Theorem (Lemma 2.1.27). We denote $K = \langle\langle H \mid (H, Y) \in \mathcal{Q}\rangle\rangle$ and $\bar{G} = G/K$. We denote by $\bar{A}$ the image of any set $A \subset G$ under the natural projection $\pi\colon G \twoheadrightarrow \bar{G}$.

The following lemma is the core of the proof of our main theorem. It brings together Theorem 2.3.8, Proposition 2.4.9 and Proposition 2.4.16.

Lemma 2.5.4. — *There exist $\beta \in (0,1)$ depending only on N with the following property. Let $U \subset G$ be a finite symmetric subset containing the identity such that $\mathrm{L}(U) \leqslant \pi \sinh 10^4\bar{\delta}$. Let $\Gamma = \langle U\rangle$. If Γ is non-elementary for the action on $\mathcal{X}$, then*

$$\omega(\bar{U}) \geqslant \beta\omega(U)$$

Proof. — We put

$$\beta = \sup_{\theta\in(0,1)} \inf\left\{\theta \cdot \frac{\log\frac{3}{2}}{\log(2c)}, 1-\theta\right\} \cdot \frac{1}{c}.$$

Let $U \subset G$ be a finite symmetric subset containing the identity such that $\mathrm{L}(U) \leqslant \pi \sinh 10^4\bar{\delta}$. Let $\Gamma = \langle U\rangle$ and assume that Γ is non-elementary for the action on $\mathcal{X}$. We are going to choose a power of U and apply Theorem 2.3.8 to that power for the

(δ_0, N)-acylindrical action of G on the $\sigma\delta$-hyperbolic space $\mathcal{X}$. By assumption, we have

$$10^4 \cdot 200\delta_0 \leqslant \pi \sinh 10^4\bar{\delta}.$$

Since Γ is non-elementary, it follows from Lemma 2.5.1 that there exists $n \geqslant 1$ depending on U such that

$$10^4 \cdot 200\delta_0 < \mathrm{L}(U^n) \leqslant 2\pi \sinh 10^4\bar{\delta}. \tag{2.5.1}$$

Let $\Gamma' = \langle U^n \rangle$. Since U is symmetric and contains the identity, $U \subset U^n$. Therefore $\Gamma = \Gamma'$. The fact that Γ is non-elementary now implies that Γ' is non-elementary. Let $p \in \mathcal{X}$ be a point almost-minimizing the ℓ^∞-energy $\mathrm{L}(U^n)$. It follows from Theorem 2.3.8 that there exist a subset $S \subset G$ such that

(i) $S \subset U^{cn}$,

(ii) $|S| \geqslant \frac{1}{c}|U^n|$,

(iii) S is $200\delta_0$-reduced at p.

We are going to estimate $\omega(\bar{U})$. Let $r \geqslant 1$. Since U is symmetric and contains the identity, (i) implies

$$B_S(r) \subset U^{cnr}.$$

Let $F(\tau_2)$ be the set of τ_2-shortening-free words associated to U and $\mathcal{Q}$. We have

$$|\bar{U}^{cnr}| \geqslant |\bar{B}_S(r)| \geqslant |\bar{F}(\tau_0) \cap \bar{B}_S(r)|.$$

Further,

$$\mathrm{L}(S, p) \leqslant \mathrm{L}(U^{cn}, p) \leqslant c\,\mathrm{L}(U^n, p) \leqslant L_0,$$

where the first inequality is (i) and the second one is the triangle inequality. The third one is due to the upper bound of Equation 2.5.1, together with the fact that the point p is almost-minimizing the ℓ^∞-energy $\mathrm{L}(U)$. Hence we can apply Proposition 2.4.9 and Proposition 2.4.16 to obtain, respectively

$$|\bar{F}(\tau_2) \cap \bar{B}_S(r)| = |F(\tau_2) \cap B_S(r)|, \quad \text{and} \quad |F(\tau_2) \cap B_S(r)| \geqslant \left[\frac{1}{2}(2|S| - 1)\right]^r.$$

Applying Fekete's Subadditive Lemma,

$$|U^n| \geqslant e^{n\omega(U)}.$$

Together with (ii), this implies

$$2|S| - 1 \geqslant |S| \geqslant \frac{1}{c} e^{n\omega(U)}.$$

Combining our estimations, we deduce

$$|\bar{U}^{cnr}| \geqslant \max\left\{ \left[\frac{1}{2}(2|S|-1)\right]^r, \left[\frac{1}{2c} e^{n\omega(U)}\right]^r \right\}. \tag{2.5.2}$$

We have,

$$\omega(\bar{U}) = \limsup_{r\to\infty} \frac{1}{cnr} \log |\bar{U}^{cnr}|.$$

Let $\theta \in (0,1)$. Consider the positive number

$$\gamma = \frac{\log 2c}{\theta\omega(U)}.$$

► If $n \leqslant \gamma$, we use the first bound of Equation 2.5.2 to obtain

$$\omega(\bar{U}) \geqslant \frac{1}{cn} \cdot \log \left[\frac{1}{2}(2|S|-1)\right].$$

Since $n \leqslant \gamma$, we have $\frac{1}{n} \geqslant \frac{1}{\gamma}$. Further, $|S| \geqslant 2$. Consequently,

$$\omega(\bar{U}) \geqslant \theta \cdot \frac{\log \frac{3}{2}}{\log 2c} \cdot \frac{1}{c} \cdot \omega(U).$$

► If $n \geqslant \gamma$, we use the second bound of Equation 2.5.2 to obtain

$$\omega(\bar{U}) \geqslant \frac{1}{c}\Big(\omega(U) - \frac{1}{n}\log 2c\Big).$$

Since $n \geqslant \gamma$, we have $\frac{1}{n} \leqslant \frac{1}{\gamma}$. Consequently,

$$\frac{1}{n} \log 2c \leqslant \theta\omega(U).$$

Therefore,

$$\omega(\bar{U}) \geqslant (1-\theta) \cdot \frac{1}{c} \cdot \omega(U).$$

Finally, combining the cases $n \leqslant \gamma$ and $n \geqslant \gamma$, we obtain:

$$\omega(\bar{U}) \geqslant \beta\omega(U).$$

$\square$

Theorem 2.5.5 (Theorem 0.6.2 (i))**.** — *Let $\xi > 0$. If G has ξ-uniform uniform exponential growth, then every geometric $C''(\lambda, \varepsilon)$-small cancellation quotient of G has ξ'-uniform uniform exponential growth. The constant ξ' depends only on ξ and N.*

Proof. — Let $\xi > 0$. Assume that G has ξ-uniform uniform exponential growth. Let $\bar{U} \subset \bar{G}$ be a finite symmetric subset containing the identity and denote $\bar{\Gamma} = \langle \bar{U} \rangle$. Recall that $\mathscr{V}$ stands by the set of apices of the cone-off space $\dot{\mathcal{X}}_\rho(\mathscr{Q}, X)$. There are two cases:

Case 1. *There exist $\bar{v} \in \bar{\mathscr{V}}$ such that $\bar{U}$ is contained in* $\mathrm{Stab}(\bar{v})$.

Let $v \in \mathscr{V}$ be a preimage of $\bar{v}$. Let $(H, Y) \in \mathscr{Q}$ such that v is the apex of the cone $Z(Y)$. The natural projection $\pi\colon G \twoheadrightarrow \bar{G}$ induces an isomorphism $\mathrm{Stab}(Y)/H \xrightarrow{\sim} \mathrm{Stab}(\bar{v})$ (Lemma 2.1.27 (iii)). Since the moving family $\mathscr{Q}$ is loxodromic, H has finite index in $\mathrm{Stab}(Y)$. Hence $\bar{\Gamma}$ is finite, in particular virtually nilpotent.

Case 2. *The set $\bar{U}$ is not contained in* $\mathrm{Stab}(\bar{v})$*, for every $\bar{v} \in \bar{\mathscr{V}}$.*

The quotient space $\bar{\mathcal{X}}_\rho$ is $\bar{\delta}$-hyperbolic (Lemma 2.1.27 (i)) and the action of $\bar{\Gamma}$ on $\bar{\mathcal{X}}_\rho$ is acylindrical (Lemma 2.1.35). Then $\bar{\Gamma}$ falls exactly in one of the following three situations (Lemma 2.1.22):

(a) *$\bar{\Gamma}$ is elliptic, or equivalently one (hence any) orbit of $\bar{\Gamma}$ is bounded.* Since the set $\bar{U}$ is not contained in $\mathrm{Stab}(\bar{v})$, for every $\bar{v} \in \bar{\mathscr{V}}$, there exists an elliptic subgroup $E \subset G$ for the action of G on $\mathcal{X}$ such that the natural projection $\pi\colon G \twoheadrightarrow \bar{G}$ induces an isomorphism $E \xrightarrow{\sim} \bar{\Gamma}$ (Lemma 2.1.31). Since G has ξ-uniform uniform exponential growth, the subgroup E is either virtually nilpotent or has ξ-uniform exponential growth. In combination with the isomorphism $F \xrightarrow{\sim} \bar{\Gamma}$, we deduce that $\bar{\Gamma}$ either is virtually nilpotent or has ξ-uniform exponential growth.

(b) *$\bar{\Gamma}$ is loxodromic, or equivalently $\bar{\Gamma}$ is virtually cyclic and contains a loxodromic element.* Then $\bar{\Gamma}$ is virtually nilpotent.

(c) $\bar{\Gamma}$ *is non-elementary, or equivalently* $\bar{\Gamma}$ *contains a free group* $\mathbf{F}_2$ *of rank* 2 *and one (hence any) orbit of* $\mathbf{F}_2$ *is unbounded.* There are two subcases:

(E1) *Large energy*: $L(\bar{U}) > 10^4\bar{\delta}$.

Then $\omega(\bar{U}) \geqslant \frac{1}{10^3}\log 2$ (Lemma 2.1.22 and Lemma 2.1.23). Note that here we do not require any control over the parameters of the acylindrical action of $\bar{\Gamma}$ on $\bar{\mathcal{X}}_\rho$.

(E2) *Small energy*: $L(\bar{U}) \leqslant 10^4\bar{\delta}$.

Since $\bar{U}$ is not contained in $\mathrm{Stab}(\bar{v})$, for every $\bar{v} \in \bar{\mathscr{V}}$, and $10^4\bar{\delta} \leqslant \rho/5$, there exists a pre-image $U \subset G$ of $\bar{U}$ of energy $L(U) \leqslant \pi \sinh 10^4\bar{\delta}$ (Lemma 2.1.32). Without loss of generality, we may assume that U is symmetric and contains the identity. Since $\bar{\Gamma}$ is non-elementary for the action on $\bar{\mathcal{X}}_\rho$, the subgroup Γ is non-elementary for the action on $\mathcal{X}$ (Lemma 2.1.29). According to Lemma 2.5.4, there exists $\beta \in (0,1)$ depending on N such that $\omega(\bar{U}) \geqslant \beta\omega(U)$. Since G has ξ-uniform uniform exponential growth and Γ is non-elementary, we have $\omega(U) \geqslant \xi$. Therefore, $\omega(\bar{U}) \geqslant \beta\xi$. This completes the proof of our theorem.

□

Theorem 2.5.6 (Theorem 0.6.2 (ii))**. —** *Let* $\xi > 0$*. If there exists a geometric* $C''(\lambda, \varepsilon)$*-small cancellation quotient of* G *that has* ξ*-uniform uniform exponential growth, then* G *has* ξ'*-uniform uniform exponential growth. The constant* ξ' *depends only on* ξ.

Proof. — Let $\xi > 0$. Assume that $\bar{G}$ has ξ-uniform uniform exponential growth. Let $U \subset G$ be a finite symmetric subset containing the identity and denote $\Gamma = \langle U \rangle$. Then Γ falls exactly in one of the following three situations (Lemma 2.1.22):

(a) Γ *is elliptic, or equivalently one (hence any) orbit of* Γ *is bounded.* The projection $\pi \colon G \twoheadrightarrow \bar{G}$ induces an isomorphism $\Gamma \xrightarrow{\sim} \bar{\Gamma}$ (Lemma 2.1.30). Since $\bar{G}$ has ξ-uniform uniform exponential growth, the subgroup $\bar{\Gamma}$ is either virtually nilpotent or has ξ-uniform exponential growth. In combination with the isomorphism $\Gamma \xrightarrow{\sim} \bar{\Gamma}$, we deduce that Γ is either virtually nilpotent or has ξ-uniform exponential growth.

(b) Γ *is loxodromic, or equivalently* Γ *is virtually cyclic and contains a loxodromic element.* Then Γ is virtually nilpotent.

(c) Γ *is non-elementary, or equivalently* Γ *contains a free group* $\mathbf{F}_2$ *of rank* 2 *and one (hence any) orbit of* $\mathbf{F}_2$ *is unbounded.* There are two subcases:

(E1) *Large energy*: $\mathrm{L}(U) > 10^4\delta_0$.

Then $\omega(U) \geqslant \frac{1}{10^3}\log 2$ (Lemma 2.1.22 and Lemma 2.1.23). Note that here we do not require any control over the parameters of the acylindrical action of Γ on $\mathcal{X}$.

(E2) *Small energy*: $\mathrm{L}(U) \leqslant 10^4\delta_0$.

By definition, $\omega(U) \geqslant \omega(\bar{U})$. Since Γ is non-elementary for its action on $\mathcal{X}$, we have $\omega(U) > 0$. Since $10^4\delta_0 \leqslant \pi \sinh 10^4\bar{\delta}$, it follows from Lemma 2.5.4 that $\omega(\bar{U}) > 0$. In particular $\bar{\Gamma}$ is not virtually nilpotent. Since $\bar{G}$ has ξ-uniform uniform exponential growth, we deduce that $\omega(\bar{U}) \geqslant \xi$. Therefore, $\omega(U) \geqslant \xi$.

□

Appendix A

Properties of constricting subsets

Contents

In this section, we fix constants $\mu \geqslant 1$, $\nu \geqslant 0$ and a (μ, ν)-path system space $(X, \mathscr{P})$. We will be looking closely at the geometric features of the constricting subsets of X.

A.1 Standard properties

The goal of this subsection is to bring together the essential properties of constricting maps that can be deduced from the definition.

Proposition A.1.1. — *For every $\delta \geqslant 0$, there exist a constant $\theta \geqslant 0$ and a pair of maps, $\sigma\colon \mathbf{R}_{\geqslant 1} \times \mathbf{R}_{\geqslant 0} \to \mathbf{R}_{\geqslant 0}$ and $\zeta\colon \mathbf{R}_{\geqslant 0} \to \mathbf{R}_{\geqslant 0}$, such that any δ-constricting map $\pi_A\colon X \to A$ satisfies the following properties:*

(1) ***Coarse nearest-point projection.***

For every $x \in X$, we have $|x - \pi_A(x)| \leqslant \mu d(x, A) + \theta$.

(2) ***Coarse gate map.***

Let $x \in X$ and $a \in A$. Let $\gamma \in \mathscr{P}$ joining x to a. If a' is the entrance point of γ on $A^{+\delta}$, then $|a' - \pi_A(x)| \leqslant \theta$.

(3) ***Coarse equivariance.***

Let G be a group acting by isometries on A such that $\mathscr{P}$ is G-invariant. Then for every $g \in G$ and for every $x \in X$, we have $|\pi_A(gx) - g\pi_A(x)| \leqslant \theta$.

(4) ***Coarse Lipschitz map.***

For every $x, y \in X$, we have $|x - y|_A \leqslant \mu |x - y| + \theta$.

(5) ***Intersection–Image.***

For every $\gamma \in \mathscr{P}$, we have $|\operatorname{diam}(A^{+\delta} \cap \gamma) - \operatorname{diam}_A(\gamma)| \leqslant \theta$.

(6) ***Behrstock inequality.***

Let $\pi_B \colon X \to B$ be a δ-constricting map. Then for every $x \in X$, we have $\min\{d_A(x, B), d_B(x, A)\}$ θ.

(7) ***Tight contraction.***

Let $x \in X$ and $r = \frac{1}{\mu} d(x, A)$. Then $\operatorname{diam}_A(B_X(x, r)) \leqslant \theta$.

(8) ***Morseness.***

Let $\kappa \geqslant 1$, $l \geqslant 0$. Let α be a (κ, l)-quasi-geodesic of X with endpoints in A. Then $\alpha \subset A^{+\sigma(\kappa, l)}$.

(9) ***Coarse invariance.***

Let $\varepsilon \geqslant 0$. Let $B \subset X$ be a subset such that $d_{\mathrm{Haus}}(A, B) \leqslant \varepsilon$. Then B is $\zeta(\varepsilon)$-constricting.

(10) ***Coarse uniqueness.***

Let $\varepsilon \geqslant 0$. Let $\pi_B \colon X \to B$ be a δ-constricting map such that $d_{\mathrm{Haus}}(A, B) \leqslant \varepsilon$. Then for every $x \in X$, we have $|\pi_A(x) - \pi_B(x)| \leqslant \zeta(\varepsilon)$.

Proof. — Our proves are based on the sketches of the following references. For (1), (4), (5) and (7), see [74, Lemma 2.4]. For (2), see [74, Lemma 5.2 (3)], [74, Lemma 5.2 (3)]. For (6), see [74, Lemma 2.5]. For (8), see [74, Lemma 2.8 (1)].

Let $\delta \geqslant 0$. We put $\theta = \max_{1 \leqslant i \leqslant 7} \theta_i$ and for every $\varepsilon \geqslant 0$, we put $\zeta(\varepsilon) = \max_{9 \leqslant i \leqslant 10} \zeta_i(\varepsilon) \geqslant 0$, where $\theta_i \geqslant 0$ and $\zeta_i(\varepsilon) \geqslant 0$ are constants whose exact values will be precised below, with the exception of θ_6, which is the constant of Proposition A.2.1. Let $\pi_A \colon X \to A$ be a δ-constricting map.

(1) Let $x \in X$. Let $a \in A$ such that $|x - a| \leqslant d(x, A) + 1$. Assume first that $|x - a| \leqslant \delta$. By the triangle inequality,

$$|x - \pi_A(x)| \leqslant |x - a| + |a - \pi_A(a)| + |a - x|_A.$$

By (CS1), we have $|a - \pi_A(a)| \leqslant \delta$. Thus,

$$|x - \pi_A(x)| \leqslant d(x, A) + 2\delta + 1.$$

Assume now that $|x-a|_A > \delta$. Let $\gamma \in \mathscr{P}$ joining x to a. By (CS2), there exists a point p in γ such that $|p - \pi_A(x)| \leqslant \delta$. By the triangle inequality,

$$|x - \pi_A(x)| \leqslant |x - p| + |p - \pi_A(x)|.$$

Since γ is a (μ, ν)-quasi-geodesic,

$$|x - p| \leqslant \mu|x - a| + \nu.$$

In conclusion,

$$|x - \pi_A(x)| \leqslant \mu d(x, A) + \mu + \nu + \delta.$$

Finally, we put $\theta_1 = \max\{2\delta + 1, \mu + \nu + \delta\}$.

(2) Let $x \in X$ and $a \in A$. Let $\gamma \in \mathscr{P}$ joining x to a. Let a' be the entrance point of γ on $A^{+\delta}$. Assume first that $|x - a'|_A \leqslant \delta$. By the triangle inequality,

$$|\pi_A(x) - a'| \leqslant |x - a'|_A + |\pi_A(a') - a'|.$$

It follows from (1) that $|\pi_A(a') - a'| \leqslant \mu\delta + \theta_1$. Consequently,

$$|\pi_A(x) - a'| \leqslant \delta + \mu\delta + \theta_1.$$

Assume now that $|x - a'|_A > \delta$. Since $[x, a']_\gamma \in \mathscr{P}$, it follows from (CS2) that there exists a point p in $[x, a']_\gamma$ such that $|\pi_A(x) - p| \leqslant \delta$. By definition of a', we have $p = a'$ and hence $|\pi_A(x) - a'| \leqslant \delta$. Finally, we put $\theta_2 = \max\{\delta + \mu\delta + \theta_1, \delta\}$.

(3) Let G be a group acting by isometries on A such that $\mathscr{P}$ is G-invariant. Let $g \in G$ and $x \in X$. Let $\gamma \in \mathscr{P}$ joining x to $\pi_A(x)$. Let a' be the entrance point of γ on $A^{+\delta}$. By the triangle inequality,

$$|\pi_A(gx) - g\pi_A(x)| \leqslant |\pi_A(gx) - ga'| + |ga' - g\pi_A(x)|.$$

Since A is G-invariant, the element ga' is the entrance point of $g\gamma$ on $A^{+\delta}$. Since $\mathscr{P}$ is G-invariant, the path $g\gamma$ belongs to $\mathscr{P}$. It follows from (2) that

$$\max\{|\pi_A(gx) - ga'|, |ga' - g\pi_A(x)|\} \leqslant \theta_2.$$

Consequently,

$$|\pi_A(gx) - g\pi_A(x)| \leqslant 2\theta_2.$$

Finally, we put $\theta_3 = 2\theta_2$.

(4) Let $x, y \in X$. It suffices to assume that $|x - y|_A > \delta$. Let $\gamma \in \mathscr{P}$ joining x to y. By (CS2), there exist $p, q \in \gamma$ such that

$$\max\{|\pi_A(x) - p|, |\pi_A(y) - q|\} \leqslant \delta.$$

By the triangle inequality,

$$|x - y|_A \leqslant |\pi_A(x) - p| + |p - q| + |q - \pi_A(y)|.$$

Since γ is a (μ, ν)-quasi-geodesic,

$$|p - q| \leqslant \mu|x - y| + \nu.$$

Consequently,

$$|x - y|_A \leqslant \mu|x - y| + 2\delta + \nu.$$

Finally, we put $\theta_4 = 2\delta + \nu$.

(5) Let $\gamma \in \mathscr{P}$. First we prove that $\operatorname{diam}_A(\gamma) \leqslant \operatorname{diam}(A^{+\delta} \cap \gamma) + \theta_5$. Let $x, y \in \gamma$. It suffices to assume that $|x - y|_A > \delta$. Since $[x, y]_\gamma \in \mathscr{P}$, there exist $p, q \in [x, y]_\gamma$ such that

$$\max\{|\pi_A(x) - p|, |\pi_A(y) - q|\} \leqslant \delta.$$

By the triangle inequality,

$$|x - y|_A \leqslant |\pi_A(x) - p| + |p - q| + |q - \pi_A(y)|.$$

Since $p, q \in A^{+\delta} \cap \gamma$, we have $|p - q| \leqslant \operatorname{diam}(A^{+\delta} \cap \gamma)$. Hence,

$$|x - y|_A \leqslant \operatorname{diam}(A^{+\delta} \cap \gamma) + 2\delta.$$

Now we prove that $\operatorname{diam}(A^{+\delta} \cap \gamma) \leqslant \operatorname{diam}_A(\gamma) + \theta_5$. Let $x, y \in A^{+\delta} \cap \gamma$. By the triangle inequality,

$$|x-y| \leqslant |x-\pi_A(x)| + |x-y|_A + |\pi_A(y)-y|.$$

By (1),

$$\max\{|\pi_A(x)-x|, |\pi_A(y)-y|\} \leqslant \mu\delta + \theta_1.$$

Since $\pi_A(x), \pi_A(y) \in \pi_A(\gamma)$, we have $|x-y|_A \leqslant \operatorname{diam}_A(\gamma)$. Hence,

$$|x-y| \leqslant \operatorname{diam}_A(\gamma) + 2\mu\delta + 2\theta_1.$$

Finally, we put $\theta_5 = \max\{2\delta, 2\mu\delta + 2\theta_1\}$.

(6) We refer to Proposition A.2.1 for this proof.

(7) Let $x \in X$ and $r = \frac{1}{\mu} d(x, A)$. It suffices to prove that for every $y \in X$, if $|x-y| \leqslant r$, then $|x-y|_A \leqslant 3\delta + \nu$. We argue by contraposition. Let $y \in X$ such that $|x-y|_A > 3\delta + \nu$. Let $\gamma\colon [0, L] \to X$ be a path of $\mathscr{P}$ joining x to y. Since γ is a (μ, ν)-quasi-geodesic,

$$|x-y| \geqslant \frac{1}{\mu} L - \frac{1}{\mu}\nu.$$

By (CS2), there exist $p, q \in \gamma$ such that

$$\max\{|\pi_A(x)-p|, |\pi_A(y)-q|\} \leqslant \delta.$$

Let $s, t \in [0, L]$ such that $p = \gamma(s)$ and $q = \gamma(t)$. We note that

$$L \geqslant \max\{s, t\} \geqslant \min\{s, t\} + |s-t|.$$

Thus,

$$|x-y| \geqslant \frac{1}{\mu}\min\{s, t\} + \frac{1}{\mu}|s-t| - \frac{1}{\mu}\nu.$$

By the triangle inequality,

$$s \geqslant |x-p| \geqslant |x-\pi_A(x)| - |\pi_A(x)-p|,$$
$$t \geqslant |x-q| \geqslant |x-\pi_A(y)| - |\pi_A(y)-q|,$$
$$|s-t| \geqslant |p-q| \geqslant |x-y|_A - |p-\pi_A(x)| - |q-\pi_A(y)|.$$

In particular,

$$\min\{s,t\} \geqslant d(x,A) - \delta, \quad |s-t| > \delta + \nu.$$

Therefore,

$$|x-y| > \frac{1}{\mu} d(x,A).$$

Finally, we put $\theta_7 = 2(3\delta + \nu)$.

(8) We refer to Proposition A.3.4 for this proof.

(9) For every $\varepsilon \geqslant 0$, we put $\zeta_9(\varepsilon) = \delta + 2(\varepsilon + 2)$. Let $\varepsilon \geqslant 0$. Let $B \subset X$ be a subset such that $d_{\mathrm{Haus}}(A,B) \leqslant \varepsilon$. We define a map $\pi_B \colon X \to B$ as follows. Since $A \subset B^{+\varepsilon+1}$, for every $x \in X$, there exists $b \in B$ such that $|b - \pi_A(x)| \leqslant \varepsilon + 2$. We put $\pi_B(x) = b$. We prove that the map $\pi_B \colon X \to B$ is ζ_9-constricting. Let $x \in B$. By the triangle inequality,

$$|\pi_B(x) - x| \leqslant |\pi_B(x) - \pi_A(x)| + |\pi_A(x) - x|.$$

By (CS1), we have $|\pi_A(x) - x| \leqslant \delta$. Therefore, we obtain $|\pi_B(x) - x| \leqslant \zeta_9$. This establishes (CS1). Let $y, z \in X$ such that $|y-z|_B > \zeta_9$. Let $\gamma \in \mathscr{P}$ joining y to z. By the triangle inequality,

$$|y-z|_A \geqslant |y-z|_B - |\pi_B(y) - \pi_A(y)| - |\pi_B(z) - \pi_A(z)|.$$

Consequently, we have $|y-z|_A > \delta$. Therefore, it follows from (CS2) that there exist $p, q \in \gamma$ such that $\max\{|\pi_A(y) - p|, |\pi_A(z) - q|\} \leqslant \delta$. By the triangle inequality,

$$|\pi_B(y) - p| \leqslant |\pi_B(y) - \pi_A(y)| + |\pi_A(y) - p|.$$

Therefore, we have $|\pi_B(y) - p| \leqslant \zeta_9$. By symmetry, we obtain $|\pi_B(z) - q| \leqslant \zeta_9$. This establishes (CS2).

(10) Let $\varepsilon \geqslant 0$. Let $\pi_B \colon X \to B$ be a δ-constricting map such that $d_{\mathrm{Haus}}(A,B) \leqslant \varepsilon$. Let $x \in X$. We bound $|\pi_A(x) - \pi_B(x)|$. Let $\gamma \in \mathscr{P}$ joining x to $\pi_A(x)$. Let a' be the entrance point of γ on $A^{+\varepsilon+1+\delta}$. By the triangle inequality,

$$|\pi_A(x) - \pi_B(x)| \leqslant |x - a'|_A + |\pi_A(a') - a'| + |a' - \pi_B(a')| + |a' - x|_B.$$

Since $a' \in A^{+\varepsilon+1+\delta}$ and $A^{+\varepsilon+1+\delta} \subset B^{+2\varepsilon+2+\delta}$, it follows from (1) that

$$\max\{|\pi_A(a') - a'|, |a' - \pi_B(a')|\} \leqslant \theta_1(2\varepsilon + 2 + \delta) + \theta_1.$$

Applying now (5) we obtain,

$$\max\{|x - a'|_A, |a' - x|_B\} \leqslant \max\{\operatorname{diam}(A^{+\delta} \cap [x, a']_\gamma), \operatorname{diam}(B^{+\delta} \cap [x, a']_\gamma)\} + \theta_5.$$

Since $A^{+\delta}$, $B^{+\delta} \subset A^{+\varepsilon+1+\delta}$ and since $[x, a']_\gamma \cap A^{+\varepsilon+1+\delta} = \{a'\}$,

$$\max\{|x - a'|_A, |a' - x|_B\} \leqslant \theta_5.$$

Therefore, we have

$$|\pi_A(x) - \pi_B(x)| \leqslant 2\theta_5 + 2\theta_1(2\varepsilon + 2 + \delta) + 2\theta_1.$$

Finally, we put $\zeta_{10}(\varepsilon) = 2\theta_5 + 2\theta_1(2\varepsilon + 2 + \delta) + 2\theta_1$.

□

A.2 Behrstock inequality

The goal of this subsection is to introduce a variant of Behrstock inequality, [11], in the context of Masur-Minsky subsurface projections.

Proposition A.2.1 ([74, Lemma 2.5]). — *For every $\delta \geqslant 0$, there exists $\theta \geqslant 0$ satisfying the following. Let $\pi_A \colon X \to A$ and $\pi_B \colon X \to B$ be δ-constricting maps. Then for every $x \in X$,*

$$\min\{d_A(x, B), d_B(x, A)\} \leqslant \theta.$$

Remark A.2.2. — The idea is that if $d_A(x, B)$ is large then A is "between" x and B.

Proof. — Let $\delta \geqslant 0$. Let $\theta_0 = \theta_0(\delta) \geqslant 0$ be the constant of Proposition A.1.1. Let $\theta > \theta_0 + 1$. Its exact value will be precised below. Let $\pi_A \colon X \to A$ and $\pi_B \colon X \to B$ be δ-constricting maps. Let $x \in X$. By symmetry, it suffices to show that if $d_A(x, B) > \theta$, then $d_B(x, A) \leqslant \theta$. Assume that $d_A(x, B) > \theta$. Let $b \in B$ and consider a path $\gamma \in \mathscr{P}$ joining x to b.

Claim A.2.3. — $A^{+\delta} \cap \gamma \neq \varnothing$.

By Proposition A.1.1 (5) *Intersection–Image*, $\operatorname{diam}(A^{+\delta} \cap \gamma) \geqslant \operatorname{diam}_A(\gamma) - \theta_0$. Moreover, $\operatorname{diam}_A(\gamma) \geqslant |x-b|_A \geqslant d_A(x,B)$. Since $d_A(x,B) > \theta_0 + 1$, we obtain $\operatorname{diam}(A^{+\delta} \cap \gamma) > 0$. This proves the claim.

Since $A^{+\delta} \cap \gamma \neq \varnothing$ we can consider the entrance point a' of γ on $A^{+\delta}$.

Claim A.2.4. — $B^{+\delta} \cap [x, a']_\gamma = \varnothing$.

To argue by contradiction, assume that there exists $y \in B^{+\delta} \cap [x, a']_\gamma$. In particular, there exists $b' \in B$ such that $|y - b'| \leqslant \delta + 1$. By the triangle inequality,

$$d_A(x,B) \leqslant |x - b'|_A \leqslant |x-y|_A + |y-b'|_A.$$

Since $[x,a']_\gamma \in \mathscr{P}$ and $A^{+\delta} \cap [x,a']_\gamma = \{a'\}$, it follows from Proposition A.1.1 (5) *Intersection–Image* that

$$|x-y|_A \leqslant \operatorname{diam}_A([x,a']_\gamma) \leqslant \operatorname{diam}(A^{+\delta} \cap [x,a']_\gamma) + \theta_0 \leqslant \theta_0.$$

By Proposition A.1.1 (4) *Coarse Lipschitz map*,

$$|y-b'|_A \leqslant \mu(\delta+1) + \theta_0.$$

Hence $d_A(x,B) \leqslant \theta$. Contradiction. Therefore $B^{+\delta} \cap [x,a']_\gamma = \varnothing$. This proves the claim.

Finally, we estimate $d_B(x,A)$. Let $a \in A$. By the triangle inequality,

$$d_B(x,A) \leqslant |x - \pi_A(x)|_B \leqslant |x-a'|_B + |a' - \pi_A(x)|_B.$$

Since $B^{+\delta} \cap [x,a']_\gamma = \varnothing$, it follows from Proposition A.1.1 (5) *Intersection–Image* that

$$|x-a'|_B \leqslant \operatorname{diam}_B([x,a']_\gamma) \leqslant \operatorname{diam}(B^{+\delta} \cap [x,a']_\gamma) + \theta_0 \leqslant \theta_0.$$

Applying together Proposition A.1.1 (2) *Coarse gate map* and (4) *Coarse Lipschitz map*, we have $|a' - \pi_A(x)|_B \leqslant \mu\theta_0 + \theta_0$. Consequently, we obtain $d_B(x,A) \leqslant \theta$ for $\theta = \max\{\theta_0 + 1, \mu(\delta+1) + 2\theta_0, 2\theta_0 + \mu\theta_0\}$. □

A.3 Morseness

There is a large number of different notions of convexity that coincide with quasi-convexity in hyperbolic spaces but differ in more general metric spaces. One of them is the notion of Morseness.

Definition A.3.1 (Morseness). — Let $\sigma\colon \mathbf{R}_{\geqslant 1} \times \mathbf{R}_{\geqslant 0} \to \mathbf{R}_{\geqslant 0}$. A *subset* $Y \subset X$ is σ*-Morse* if for every $\kappa \geqslant 1$, $l \geqslant 0$, any (κ, l)-quasi-geodesic of X with endpoints in Y is contained in the $\sigma(\kappa, l)$-neighbourhood of Y.

Example A.3.2. — A geodesic metric space X is hyperbolic if and only if there exists $\sigma\colon \mathbf{R}_{\geqslant 1} \times \mathbf{R}_{\geqslant 0} \to \mathbf{R}_{\geqslant 0}$ such that any geodesic segment of X is σ-Morse [61, Lemma 7.2].

The goal of this section is to show that constricting subsets of X are Morse. We would like to emphasize that it is possible to give a proof that does not involve the path system, following the argument of [74, Lemma 2.8]. For this reason, we introduce the following notion of convexity.

Definition A.3.3 (Weak contraction). — Let $M \geqslant 1$, $\Delta \geqslant 0$. A map $\pi_A\colon X \to A$ from X to a subset $A \subset X$ is (M, Δ)-*weakly contracting* if it verifies the following properties.

$(WC1)$ **Coarse nearest-point projection.**

For every $x \in X$, we have $|x - \pi_A(x)| \leqslant Md(x, A) + \Delta$.

$(WC2)$ **Contraction.**

Let $x \in X$ and $r = \frac{1}{M}d(x, A) - \Delta$. Then $\operatorname{diam}_A(B_X(x, r)) \leqslant \Delta$.

A subset $A \subset X$ is (M, Δ)-*weakly contracting* if there exists a (M, Δ)-contracting map $\pi_A\colon X \to A$.

It follows from Proposition A.1.1 (1) *Coarse nearest point projection* and Proposition A.1.1 (7) *Tight contraction* that δ-constricting subsets of X are always weakly contracting with constants depending on μ, ν, δ.

Proposition A.3.4 ([74, Lemma 2.8]). — *For every $M \geqslant 1$, $\Delta \geqslant 0$, there exists $\sigma\colon \mathbf{R}_{\geqslant 1} \times \mathbf{R}_{\geqslant 0} \to \mathbf{R}_{\geqslant 0}$ such that any (M, Δ)-weakly contracting subset $A \subset X$ is σ-Morse. In particular, constricting subsets are Morse.*

Remark A.3.5. — There exist a metric space X containing a Morse subset that is not weakly contracting, [69, Example 3.8]. In particular this Morse subset is not constricting with respect to any path system of X.

From now on in this subsection, we fix $M \geqslant 1$, $\Delta \geqslant 0$. Let $\pi_A\colon X \to A$ be a (M,Δ)-weakly contracting map. The idea of the proof consists on finding a large enough neighbourhood of A so that given a quasi-geodesic with endpoints in A, the subpaths that intersect this neighbourhood only at both endpoints have uniformly bounded length. The purpose of the following lemmas is to estimate these lengths. The neighbourhood will depend only on the rescaling constant of the quasi-geodesic.

Lemma A.3.6. — *For every $\kappa \geqslant 1$, $l \geqslant 0$, $\eta \geqslant 0$, there exists $\theta \geqslant 0$ with the following property. Let α be a (κ, l)-quasi-geodesic of X such that $\alpha \cap A^{+\eta} = \{\alpha^-, \alpha^+\}$. Then*

$$|\alpha^- - \alpha^+|_A \geqslant \frac{1}{\kappa}\ell(\alpha) - \theta.$$

Proof. — Let $\kappa \geqslant 1$, $l \geqslant 0$, $\eta \geqslant 0$. Let $\theta \geqslant 0$. Its exact value will be precised below. Let α be a (κ, l)-quasi-geodesic of X such that $\alpha \cap A^{+\eta} = \{\alpha^-, \alpha^+\}$. By the triangle inequality,

$$|\alpha^- - \alpha^+|_A \geqslant |\alpha^- - \alpha^+| - |\alpha^- - \pi_A(\alpha^-)| - |\alpha^+ - \pi_A(\alpha^+)|.$$

Since α is a (κ, l)-quasi-geodesic,

$$|\alpha^- - \alpha^+| \geqslant \frac{1}{\kappa}\ell(\alpha) - \frac{1}{\kappa}l.$$

Since $\alpha^-, \alpha^+ \in A^{+\eta}$, it follows from (WC1) that

$$\max\{|\alpha^- - \pi_A(\alpha^-)|, |\alpha^+ - \pi_A(\alpha^+)|\} \leqslant M\eta + \Delta.$$

Finally, we put $\theta = \frac{1}{\kappa}l + 2M\eta + 2\Delta$ □

Lemma A.3.7. — *For every $\kappa > 0$, there exists $\eta \geqslant 0$ with the following property. Let α be a path of X such that $d(\alpha, A) \geqslant \eta$. Then*

$$|\alpha^- - \alpha^+|_A \leqslant \frac{1}{\kappa}\ell(\alpha) + \Delta + 1.$$

Proof. — Let $\kappa > 0$. Let $\eta = M(\Delta+1)\kappa + M\Delta$. Let $\alpha\colon [0, L] \to X$ be a path such that $d(\alpha, A) \geqslant \eta$. We estimate $|\alpha^- - \alpha^+|_A$. Let $\zeta = (\Delta+1)\kappa$. Since $\zeta > 0$, we can define $m = \left\lfloor \frac{L}{\zeta} \right\rfloor + 1$. We fix a partition $0 = t_0 \leqslant t_1 \leqslant \cdots \leqslant t_m = L$ of $[0, L]$ such that $|t_{m-1} - t_m| \leqslant \zeta$ and such that if $m \geqslant 2$, then for every $i \in [\![0, m-2]\!]$, we have $|t_i - t_{i+1}| = \zeta$.

Denote $x_i = \alpha(t_i)$. By the triangle inequality,

$$|\alpha^- - \alpha^+|_A \leqslant \sum_{i=0}^{m-1} |x_i - x_{i+1}|_A.$$

Let $i \in [\![0, m-1]\!]$. Recall that, by convention, all of our paths are parametrised by arc length. Hence, $|x_i - x_{i+1}| \leqslant \zeta$. Moreover, $\zeta = \frac{1}{M}\eta - \Delta$. Consequently,

$$|x_i - x_{i+1}| \leqslant \frac{1}{M} d(x_i, A) - \Delta.$$

Denote $r_i = \frac{1}{M} d(x_i, A) - \Delta$. By (WC2),

$$|x_i - x_{i+1}|_A \leqslant \operatorname{diam}_A(B_X(x_i, r_i)) \leqslant \Delta.$$

Hence, $|x - y|_A \leqslant m(\Delta + 1)$. By construction of the partition, $m \leqslant \zeta^{-1} L + 1$. Therefore,

$$|x - y|_A \leqslant \frac{1}{\kappa} L + \Delta + 1.$$

□

We are ready to proof the proposition:

Proof of Proposition A.3.4. — Let $\kappa \geqslant 1$, $l \geqslant 0$. Let α be a (κ, l)-quasi-geodesic of X with endpoints in A. Let $\kappa_0 > \kappa$. It follows from Lemma A.3.6 and Lemma A.3.7 that there exist $\eta = \eta(\kappa_0) \geqslant 0$ and $\theta = \theta(\kappa, l, \eta) \geqslant 0$ such that for every subpath β of α satisfying $\beta \cap A^{+\eta} = \{\beta^-, \beta^+\}$, we have

$$\ell(\beta) \leqslant \left(\frac{1}{\kappa} - \frac{1}{\kappa_0}\right)^{-1} (\Delta + 1 + \theta).$$

Moreover, we can decompose α as an union of subpaths that either intersect $A^{+\eta}$ only at both endpoints or are contained in $A^{+\eta}$. This is enough to prove that there exist $\sigma\colon \mathbf{R}_{\geqslant 1} \times \mathbf{R}_{\geqslant 0} \to \mathbf{R}_{\geqslant 0}$ such that A is σ-Morse. □

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COLLEGE DOCTORAL BRETAGNE — MATHS, TELECOMS, INFORMATIQUE, SIGNAL, SYSTEMES, ELECTRONIQUE

Université de Rennes

Titre : Croissance dans les groupes à courbure négative ou nulle

Mot clés : théorie géométrique des groupes, groupes hyperboliques et leurs généralisations, croissance exponentielle, théorie de la petite simplification

Résumé : L'objectif de cette thèse est d'obtenir une meilleure compréhension du comportement des taux de croissance exponentiels au sein de la classe des groupes qui agissent de manière acylindrique dans un espace hyperbolique au sens de Gromov. Pour ce faire, nous aborderons deux problèmes de nature différente.

Dans le premier problème, nous étudierons les taux de croissance exponentiels des sous-groupes quasi-convexes. Nous comparerons ces taux avec celui du groupe ambiant et nous déterminerons quand il est possible d'obtenir une égalité/inégalité stricte. Pour ce faire, nous allons exploiter des actions propres sur des espaces métriques, a priori, non hyperboliques, mais dont les isométries se comportent comme les isométries loxodromiques d'un espace hyperbolique.

Le deuxième problème tourne autour de la croissance exponentielle uniforme uniforme. Nous prouverons que cette propriété est préservée si nous prenons des quotients à petite simplification de groupes qui agissent de manière acylindrique sur un espace hyperbolique. En corollaire, nous obtiendrons qu'il existe une borne inférieure universelle sur le taux de croissance exponentielle uniforme pour la famille des quotients à petite simplification classique. Cette borne ne dépend que d'un des deux paramètres d'acylindricité.

Title: Growth in Groups of Non-Positive Curvature

Keywords: geometric group theory, hyperbolic groups and their generalisations, exponential growth, small cancellation theory

Abstract: The aim of this thesis is to obtain a better understanding of the behavior of exponential growth rates within the class of groups that act acylindrically in a hyperbolic space in the sense of Gromov. To do this, we will address two problems of a different nature.

In the first problem we will study the exponential growth rates of quasi-convex subgroups. We will compare these rates with that of the ambient group and we will determine when it is possible to obtain strict equality/inequality. To do so, we will exploit proper actions on metric spaces that, a priori, are not hyperbolic, but that have isometries that behave like the loxodromic isometries of a hyperbolic space.

The second problem revolves around uniform uniform exponential growth. We will prove that this property is preserved if we take small cancellation quotients of groups that act acylindrically on a hyperbolic space. As a corollary, we will obtain that there is a universal lower bound on the uniform exponential growth rate for the family of classical small cancellation quotients. This bound depends only on one of the two acylindricity parameters.