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UNIFORM EXPONENTIAL GROWTH IN SMALL CANCELLATION GROUPS

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Abstract. *A leitmotiv in geometric group theory is to identify properties for large classes of groups by exploiting their actions on Gromov hyperbolic spaces. A major open question asks whether every group acting acylindrically on a hyperbolic space has uniform exponential growth. We prove that the class of groups of uniform uniform exponential growth acting acylindrically on a hyperbolic space is stable under taking geometric small cancellation quotients.*

ξ-UNIFORM UNIFORM EXP. GROWTH

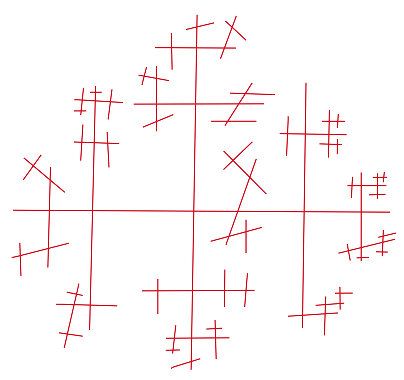
Let G be group. Let U be a finite subset. The *n-th product set* U^n is the collection of products $u_1 \cdot \ldots \cdot u_n \in G$ such that $u_1, \cdots, u_n \in U$. The *exponential growth rate of* U is the number

$$\omega(U) = \limsup_{n \to \infty} \frac{1}{n} \log |U^n|.$$

Let $\xi > 0$. We say that G has *ξ-uniform exponential growth* if it is finitely generated (f.g.) and for every finite symmetric generating set U of G, we have $\omega(U) > \xi$. We say that G has *ξ-uniform uniform exponential growth* if every f.g. subgroup is either virtually nilpotent or has ξ-uniform exponential growth.

δ-HYPERBOLIC SPACE

Let $\delta > 0$. A geodesic metric space is *δ-hyperbolic* if every geodesic triangle is δ-thin. An isometry g of X is *loxodromic* if the map that sends $n \in \mathbf{Z}$ to $g^n x$ is a quasi-isometric embedding for some $x \in X$.



A large-scale viewpoint of subset in a δ-hyperbolic space.

(κ, N)-ACYLINDRICAL ACTION

Let δ, κ, $N > 0$. The action of a group G on a δ-hyperbolic space X is *(κ, N)-acylindrical* if for every pair of points $x, y \in X$ at distance at least κ, the number of elements $u \in G$ moving each of the points x, y at distance at most 100δ is bounded above by N.

The number N has two meanings for us:

(1) The largest size of the finite subgroups of virtually cyclic subgroups in G containing a loxodromic isometry.
(2) The fraction of **a)** the longest overlap between the axis of any pair of conjugates of an arbitrary loxodromic isometry g of G, with **b)** the *translation length* of g, whenever this translation is larger than 100δ.

Example:

The following are families of groups with uniform uniform exponential growth acting acylindrically on hyperbolic spaces:

1. Hyperbolic groups.
2. Free products of *countable families* of groups with ξ-uniform uniform exponential growth.
3. Some CAT(0) cubical groups.
4. Mapping class groups.

In general, the uniform growth parameter depends on the group.

TOWARDS GEOMETRIC SMALL CANCELLATION THEORY

We prove that the class of groups of uniform uniform exponential growth acting acylindrically on a hyperbolic space is closed under taking geometric $C''(\lambda, \varepsilon)$-small cancellation quotients in the sense of [Dahmani, Guirardel & Osin; Definition 6.22]. Let G be a group acting by isometries on a δ-hyperbolic space X. A *moving family* – or *set of relations* – is a set of the form

$$\mathscr{F} = \{ \, (\langle grg^{-1} \rangle, gY_r) \quad | \quad r \in \mathscr{R}, \; g \in G \, \},$$

where $\mathscr{R} \subset G$ is a set of loxodromic isometries r – the *relators* – stabilizing their quasi-convex axis $Y_r \subset X$. A *piece* is an intersection of any pair of such axis. The role of the parameters $\lambda \in (0,1)$ and $\varepsilon > 0$ in the geometric $C''(\lambda, \varepsilon)$-small cancellation condition on $\mathscr{F}$ is the following:

- The fraction of the *length of the longest piece* with the *shortest translation length of the relators* $r \in \mathscr{R}$ is at most λ.
- The *shortest translation length of the relators* $r \in \mathscr{R}$ is at least $\varepsilon\delta$.

Let K be the normal closure in G of the *relator subgroups* H in $\mathscr{F}$. The geometric $C''(\lambda, \varepsilon)$-small cancellation condition permits to obtain substantial information of the geometric $C''(\lambda, \varepsilon)$-small cancellation quotient $\bar{G} = G/K$: for instance K is a free product of relator subgroups, $\bar{G}$ locally looks like G and any acylindrical action of G on X induces another acylindrical action of $\bar{G}$ on a quotient δ_0-hyperbolic space $\bar{X}$ whose hyperbolicity constant δ_0 is universal.

The following results are from [Legaspi & Steenbock].

Main result

There exists $\lambda \in (0,1)$ such that for every $N > 0$ and $\varepsilon > 10^{10}N$ the following holds. Let $\delta > 0$, $\kappa > 100\delta$ and let G be a group acting (κ, N)-acylindrically on a δ-hyperbolic space X.

(i) If G has ξ-uniform uniform exponential growth, then every geometric $C''(\lambda, \varepsilon)$-small cancellation quotient of G has ξ'-uniform uniform exponential growth. The constant ξ' depends only on ξ and N.
(ii) If there exist a geometric $C''(\lambda, \varepsilon)$-small cancellation quotient of G that has ξ-uniform uniform exponential growth, then G has ξ'-uniform uniform exponential growth. The constant ξ' depends only on ξ.

BEYOND SHORT LOXODROMICS, LET THERE BE MONSTERS

The standard strategy to study uniform exponential growth in hyperbolic groups exploits the fact that their finite symmetric generating sets have the *short loxodromic property*: every n-th power U^n of a finite symmetric generating set contains a loxodromic isometry, for some number n that does not depend on the set U. In general, it is unknown whether every f.g. group acting acylindrically on a hyperbolic space has uniform exponential growth. The acylindrical action on a hyperbolic space yields uniform exponential growth for finite symmetric generating sets with a long loxodromic isometry. The short loxodromic property permits to take uniform large powers so that we can exploit this other situation. However, there is a f.g. (*combinatorial/graded*) small cancellation quotient with an acylindrical action on a hyperbolic space but without the short loxodromic property, [Minasyan & Osin]. Our main result does not make use of the short loxodromic property. The moral of our work is that we can deal with this kind of monster *as long as* these are small cancellation quotients of groups of uniform uniform exponential growth acting acylindrically on a hyperbolic space. However, the aforementioned monster is a quotient of the free product of all hyperbolic groups!

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