UNIFORM UNIFORM EXPONENTIAL GROWTH IN SMALL CANCELLATION QUOTIENTS

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ABSTRACT. — A major open question asks whether every group acting acylindrically on a hyperbolic space has uniform exponential growth. We prove that the class of groups of uniform uniform exponential growth acting acylindrically on a hyperbolic space is stable under taking geometric small cancellation quotients.

Keywords. Exponential growth, hyperbolic groups, acylindrical actions, small cancellation.

MSC. 20F06, 20F65, 20F67, 20F69

Contents

1. Introduction

Let G be a group with finite symmetric generating set U. Denote by X_U the corresponding Cayley graph. The n-th product set U^n is the collection of elements $u_1 \cdot ... \cdot u_n \in G$ such that $u_1, \cdots, u_n \in U$. In this article we study the number

$$\omega(U) := \limsup_{n\to\infty} \frac{1}{n} \log |U^n|.$$

The role of $\omega(U)$ is to give us information about the exponential behaviour of $|U^n|$ as n increases. The generating sets of virtually nilpotent groups have vanishing exponential growth rate, since a celebrated theorem of M. Gromov shows that those are exactly the groups of polynomial growth, [27]. Let $\xi > 0$. The group G has *ξ-uniform exponential growth* if for every finite symmetric generating set U of G, we have $\omega(U) > \xi$. A group has *ξ-uniform uniform exponential growth* if every finitely generated subgroup is either virtually nilpotent or has ξ-uniform exponential growth.

Uniform uniform exponential growth is particularly well-studied in groups of non-positive curvature. Indeed, groups of uniform uniform exponential growth include hyperbolic groups, [4, 9, 31], free products of countable families of groups with ξ-uniform uniform exponential growth (folklore), mapping class groups, [1, 2, 32], or cocompactly special cubulated CAT(0) groups, [1, 24]. It is unknown whether the outer automorphism group of the free group of rank $\geqslant 2$ has uniform uniform exponential growth, [6]. All of the groups in this list admit non-elementary acylindrical actions on Gromov hyperbolic spaces, [7, 20, 35].

1.1. Geometric small cancellation quotients

The main goal of this article is to prove that the class of groups of uniform uniform exponential growth acting acylindrically on a hyperbolic space is closed under taking geometric $C''(\lambda, \varepsilon)$-small cancellation quotients in the sense of [20, Definition 6.22]. This result is Theorem 1.2 below. Before stating the theorem, we are going to give some definitions. Let $\delta > 0$. Let G be a group acting by isometries on a δ-hyperbolic space X.

Acylindricity. Let κ, $N > 0$. The action of G on X is (κ, N)-*acylindrical* if for every pair of points $x, y \in X$ at distance at least κ, the number of elements $u \in G$ moving each of the points x, y at distance at most 100δ is bounded above by N. In practice, the number N has two meanings for us:

(1) The largest size of the finite subgroups of virtually cyclic subgroups in G containing a loxodromic isometry.

(2) The fraction $\frac{\Delta(g)}{\|g\|}$ of the longest intersection $\Delta(g)$ between the axis of any pair of conjugates of an arbitrary loxodromic isometry g of G, with the *translation length* $\|g\|$ of g, whenever this translation is larger than 100δ.

Geometric small cancellation theory. A *loxodromic moving family* – or *set of relations* – is a set of the form

$$\mathscr{Q} = \{ \ (\langle grg^{-1} \rangle , gY_r) \quad | \quad r \in \mathscr{R}, \ g \in G \ \},$$

where $\mathscr{R} \subset G$ is a set of loxodromic isometries r – the *relators* – stabilizing their quasi-convex axis $Y_r \subset X$. A *piece* is an intersection of any pair of such axis. The role of the parameters $\lambda \in (0,1)$ and $\varepsilon > 0$ in the geometric $C''(\lambda, \varepsilon)$-small cancellation condition on $\mathscr{Q}$ is the following:

- The fraction of the *length of the longest piece* with the *shortest translation length of the relators* $r \in \mathscr{R}$ is at most λ.
- The *shortest translation length of the relators* $r \in \mathscr{R}$ is at least $\varepsilon\delta$.

Let K be the normal closure in G of the *relator subgroups* H in $\mathscr{Q}$. The geometric $C''(\lambda, \varepsilon)$-small cancellation condition permits to obtain substantial information of the geometric $C''(\lambda, \varepsilon)$-small cancellation quotient $\bar{G} = G/K$: for instance K is a free product of relator subgroups, $\bar{G}$ locally looks like G and any acylindrical action of G on X induces another acylindrical action of $\bar{G}$ on a quotient $\bar{\delta}$-hyperbolic space $\bar{X}$ whose hyperbolicity constant $\bar{\delta}$ is universal.

Main theorem. The following result captures the essence of the main theorem.

Theorem 1.1. — *There exists a universal constant $\lambda > 0$ such that for every group G acting acylindrically on a hyperbolic space X, there exist $\varepsilon > 0$ depending only on the acylindricity and hyperbolicity constants such that the following statements are equivalent.*

(i) G has uniform uniform exponential growth.

(ii) Every geometric $C''(\lambda, \varepsilon)$-small cancellation quotient of G has uniform uniform exponential growth.

(iii) *There exists a geometric $C''(\lambda, \varepsilon)$-small cancellation quotient of G that has uniform uniform exponential growth.*

The main theorem of this article is more precise:

Theorem 1.2 (Theorem 6.5 & Theorem 6.6)**. —** *There exists $\lambda > 0$ such that for every $N > 0$ the following holds. Let $\delta > 0$, $\kappa \geqslant \delta$, and $\varepsilon \geqslant 10^{10} \max\{N, \kappa/\delta\}$. Let G be a group acting (κ, N)-acylindrically on a δ-hyperbolic space X.*

(i) *If G has ξ-uniform uniform exponential growth, then every geometric $C''(\lambda, \varepsilon)$-small cancellation quotient of G has ξ'-uniform uniform exponential growth. The constant ξ' depends only on ξ and N.*

(ii) *If there exist a geometric $C''(\lambda, \varepsilon)$-small cancellation quotient of G that has ξ-uniform uniform exponential growth, then G has ξ'-uniform uniform exponential growth. The constant ξ' depends only on ξ.*

Remark 1.3. — The dependence of ε on κ, N and δ is not a strong condition. In fact, the intersection of the axis of two loxodromic elements in a group acting acylindrically on a hyperbolic space is controled in terms of κ, N, δ and the translation length of the loxodromic elements. Thus to prove that a set of relators satisfies the geometric $C''(\lambda, \varepsilon)$-condition, one usually considers relators of sufficient length compared to κ, N and δ anyway.

1.2. Beyond short loxodromics

The standard strategy to study uniform exponential growth in hyperbolic groups exploits the fact that their finite symmetric generating sets have the *short loxodromic property*: every n-th power U^n of a finite symmetric generating set contains a loxodromic isometry, for some number n that does not depend on the set U. In general, it is unknown whether every finitely generated group acting acylindrically on a hyperbolic space has uniform exponential growth. The acylindrical action on a hyperbolic space yields uniform exponential growth for finite symmetric generating sets with a long loxodromic isometry. The short loxodromic property permits to take uniform large powers so that we can exploit this other situation. However, there is a finitely generated (*combinatorial/graded*) small cancellation quotient with an acylindrical action on a hyperbolic space but without the short loxodromic property, [33]. Our main result does not make use of the short loxodromic property. The moral of our work is that we can deal with this kind of monster *as long as* these are small cancellation quotients of groups of uniform uniform exponential growth acting acylindrically on a hyperbolic space. However, the aforementioned monster is a quotient of the free product of all hyperbolic groups. It is unkown whether this free product has uniform uniform exponential growth, owing to it is unkown whether there is a universal lower bound for the uniform growth rate of all hyperbolic groups, independent of the hyperbolicity constant, [9, Section 14, Question 2]. The following example shows that the short loxodromic property plays no role in the proof of Theorem 1.2.

Example 1.4. — There are infinite families of geometric small cancellation quotients that are hyperbolic groups containing arbitrarily large torsion balls. These groups act acylindrically with uniform acylindricity parameters and have ξ-uniform uniform exponential growth, for some uniform growth exponent $\xi > 0$, see [19]. The uniform uniform exponential growth rate of the small cancellation quotient in Theorem 1.2 (i) does not depend on the cardinality of large torsion balls, nor does it depend on the hyperbolicity constant.

1.3. Classical small cancellation groups

We now discuss groups given by a presentation that satisfies the classical $C''(\lambda)$-small cancellation condition. We refer to a group admiting such a presentation as classical $C''(\lambda)$-small cancellation group. These are exacly the geometric small cancellation quotients over free groups. In this situation, the geometric small cancellation condition involving the parameter ε becomes trivial. A classical $C''(\lambda)$-small cancellation group is always finitely presented, hence, hyperbolic. Thus it has uniform uniform exponential growth by [28, 31]. However, in that approach the uniform uniform exponential growth rate depends on λ. The following is a consequence of Theorem 1.2 for the free group case.

Corollary 1.5. — *There exist $\lambda > 0$ and $\xi > 0$ such that every classical $C''(\lambda)$-small cancellation group has ξ-uniform uniform exponential growth.*

Note that there is a generic class of classical $C''(1/6)$-small cancellation groups such that every 2-generated subgroup is free, [5]. This immediately implies Corollary 1.5 for this generic class of classical $C''(1/6)$-small cancellation groups, [21].

Remark 1.6. — The classical $C''(\lambda)$-small cancellation condition in Corollary 1.5 is reminiscent of our proof that uses geometric small cancellation theory. To this date, geometric small cancellation theory has not been developed under a geometric $C'(\lambda, \varepsilon)$-small cancellation condition. We expect, however, that this is possible, and thus that our results hold for classical $C'(\lambda)$-small cancellation groups - finitely and infinitely presented.

1.4. Strategy of proof

To prove Theorem 1.2 (i), we need to discuss the growth of finite symmetric subsets of sufficiently large energy in groups acting acylindrically on a hyperbolic space X. If G acts by isometries on X, the ℓ^∞-energy $\mathrm{L}(U)$ of a finite subset $U \subset G$ is defined by

$$\mathrm{L}(U) = \inf_{x \in X} \max_{u \in U} |ux - x|.$$

If $U = \{g\}$, the ℓ^∞-energy coincides with the translation length of g. The following example explains why the energy is important in the study of uniform exponential growth.

Example 1.7. — When G is the fundamental group of a compact hyperbolic manifold, there exists a constant $\mu > 0$ – the *Margulis constant* – such that if $U \subset G$ is a finite set with $\mathrm{L}(U) < \mu$, then the subgroup of G generated by U is virtually nilpotent. If T denotes the injectivity radius of the action of G on the universal cover and is smaller than the Margulis constant μ, then the acylindricity constant κ is about $1/\mathrm{T}$, [23].

Definition 1.8 (Definition 3.1)**. —** Let $\alpha > 0$. We say that a finite subset $U \subset G$ is *α-reduced at $p \in X$* if $U \cap U^{-1} = \varnothing$ and for every pair of distinct $u_1, u_2 \in U \sqcup U^{-1}$, the Gromov product satisfies

$$(u_1p, u_2p)_p < \frac{1}{2}\min\{|u_1p - p|, |u_2p - p|\} - \alpha - 2\delta.$$

Remark 1.9. — Roughly speaking, if a set $U \subset G$ is reduced then the orbit map from the free group generated by U to X is a quasi-isometric embedding.

The following is a well-known theorem of [4,31], see also [25].

Theorem 1.10 (Theorem 4.8)**. —** *For every κ, $N > 0$, there exist an integer $c > 1$ with the following property. Let δ, $\alpha > 0$. Let G be a group acting (κ, N)-acylindrically on a δ-hyperbolic space X. Let $U \subset G$ be a finite symmetric subset containing the identity. Then one of the following conditions holds:*

(i) $\mathrm{L}(U) \leqslant 10^4 \max\{\kappa, \delta, \alpha\}$.

(ii) The subgroup $\langle U \rangle$ is virtually cyclic and contains a loxodromic element.

(iii) There exists an α-reduced subset $S \subset U^c$ such that

$$|S| \geqslant \max\left\{2, \frac{1}{c}|U|\right\}.$$

Moreover,

$$\omega(U) \geqslant \frac{1}{c}\log|U|.$$

Our main contribution to Theorem 4.8 is the dependence of the involved constants: for our purpose it is important that the number c only depends on the acylindricity parameters κ and N.

Remark 1.11. — If the injectivity radius of the action of G on X is large, then every finite symmetric subset of G satisfies either (ii) or (iii). In general this is however not the case. We will later use uniform uniform exponential growth of G in order to apply Theorem 4.8 to some power of an arbitrary symmetric subset U in G.

Theorem 4.8 with *Fekete's Subadditive Lemma* and the fact that $\omega(U^n) = n\omega(U)$ implies the following corollary. It is a weak form of *purely exponential growth*, [11,37].

Corollary 1.12. — *For every κ, $N > 0$, there exists $\xi > 1$ with the following property. Let $\delta > 0$ and $\kappa \geqslant \delta$. Let G be a group acting (κ, N)-acylindrically on a δ-hyperbolic space X. Let $U \subset G$ be a finite symmetric subset containing the identity of energy $\mathrm{L}(U) > 10^4\kappa$ that does not generate a virtually cyclic subgroup. Then, for every $n \geqslant 0$,*

$$e^{n\omega(U)} \leqslant |U^n| \leqslant e^{\xi n\omega(U)}.$$

To prove Theorem 1.2 (i), we follow a strategy of [19] that estimates product set growth in Burnside groups. In particular, we use the viewpoint of geometric small cancellation theory. As previously mentioned, the *Small Cancellation Theorem* gives a universal constant $\bar{\delta} > 0$ such that any geometric small cancellation quotient $\bar{G}$ of a group G acting acylindrically on a δ-hyperbolic space X, for appropriate choice of the small cancellation parameters, acts acylindrically on a $\bar{\delta}$-hyperbolic space $\bar{X}$. Let $\bar{U} \subset \bar{G}$ be a finite symmetric generating set containing the identity that is not contained in an elliptic or virtually cyclic subgroup. If the energy of $\bar{U}$ is larger than $10^4\bar{\delta}$, then the exponential growth rate of $\bar{U}$ is bounded below by a universal strictly positive constant (Lemma 2.23). Otherwise, we fix a pre-image U of $\bar{U}$ in G of minimal energy for the action of G on X (Lemma 2.32). Such a pre-image may not have large energy $> 10^4\delta$. Indeed, it may consist entirely of torsion-elements and thus have small energy $< 10^4\delta$. However, our pre-image U is not contained in any elliptic subgroup. Thus some power of U contains a loxodromic element, hence, for some exponent n, we have $\mathrm{L}(U^n) > 10^4\delta$. We stress that the exponent n depends on the set U. We now apply Theorem 1.10 to U^n. Since U is not contained in any virtually cyclic subgroup, we obtain a reduced subset S in U^{cn}, which freely generates a free subgroup. Next, we adapt the counting argument of [13, 19] to prove that for every $r \geqslant 1$, the proportion of elements in S^r that contain a large part of a relator is small compared to $|S^r|$ (Proposition 5.9). A combination of a consequence of *Greendlinger's Lemma* (Proposition 5.16) and *Fekete's Subadditive Lemma* then implies that the exponential growth rate of $\bar{U}$ satisfies

$$\omega(\bar{U}) \geqslant \beta \cdot \omega(U).$$

for

$$\beta = \sup_{\theta \in (0,1)} \inf \left\{ \theta \cdot \frac{\log \frac{3}{2}}{\log(2c)}, 1 - \theta \right\} \cdot \frac{1}{c}.$$

Finally, assume that G has ξ-uniform uniform exponential growth. A combination of this fact with the previous inequality yields Theorem 1.2 (i). The proof of Theorem 1.2 (ii) is similar and we postpone its discussion.

1.5. Outline of the article

In Section 2.1 we will overview Gromov hyperbolic spaces, acylindricity and geometric small cancellation theory. In section 3 we will see that reduced subsets generate free subgroups with the Geodesic Extension Property. This property will be relevant to the counting argument of subsection 5.2. In section 4 we generalise work of M. Koubi, [31], and G. Arzhantseva - I. Lysenok, [4]. The goal is to produce reduced subsets inside uniform powers of other subsets of isometries. In section 5 we study the subsets of shortening-free words of a free subgroup generated by a reduced subset. These are infinite subsets, each depending on a geometric small cancellation family, such that (i) their elements are not killed when taking the geometric small cancellation quotient and (ii) their relative growth rate does not decrease too much when taking the geometric small cancellation quotient. We will prove (i) and (ii) in subsection 5.2 and subsection 5.3, respectively. Finally, Section 5 is devoted to the proof of our main theorem (Theorem 1.2).

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2. Hyperbolic geometry

We collect some facts on hyperbolic geometry in the sense of Gromov, [28], including its version of small cancellation theory, [22, 29]. See also [12, 16, 26, 30].

2.1. Hyperbolicity

Let X be a metric space. Given two points x, $x' \in X$ we write $|x - x'|$ for the distance between them and $[x, x']$ for a geodesic joining them. Recall that there may be multiple geodesics joining two points. If $Y \subset X$ is a subset and $x \in X$ a point, we write $d(x, Y) = \inf_{y \in Y} |x - y|$ to denote the *distance from x to Y*. Given $\varepsilon \geqslant 0$, we let $Y^{+\varepsilon} = \{\, x \in X \,:\, d(x, Y) \leqslant \varepsilon \,\}$ be the *ε-neighbourhood of Y*. The *Gromov product* of three points x, y, $z \in X$ is defined by

$$(x, y)_z = \frac{1}{2}\{|x - z| + |y - z| - |x - y|\}.$$

Definition 2.1. — Let $\delta \geqslant 0$. The metric space X is *δ-hyperbolic* if it is geodesic and for every x, y, z and $t \in X$, the *four point inequality* holds, that is

$$(x, z)_t \geqslant \min\{(x, y)_t, (y, z)_t\} - \delta.$$

Convention 2.2. — Let $\delta \geqslant 0$. For the remainder of this section, we assume that the space X is δ-hyperbolic. If $\delta = 0$, then it can be isometrically embedded in an $\mathbf{R}$-tree, [26, Chapitre 2, Proposition 6]. Note that X is δ'-hyperbolic for every $\delta' \geqslant \delta$. *In this chapter we always assume for convenience that the hyperbolicity constant δ is positive.*

We write ∂X for the Gromov boundary of X. We can use the boundary defined with sequences converging at infinity, [12, Chapitre 2, Définition 1.1]. Note that we did not assume the space X to be proper, thus we use the boundary defined with sequences converging at infinity, [12, Chapitre 2, Définition 1.1]. Hyperbolicity has the following consequences.

Lemma 2.3 ([23, Lemmas 2.3 and 2.4]). — *Let $x, y, z \in X$. Then*

$$(x, y)_z \leqslant d(z, [x, y]) \leqslant (x, y)_z + 4\delta.$$

Lemma 2.4 ([4, Lemma 2]). — *Let $i \in [\![1, 2]\!]$. Let x_i, $y_i \in X$. Then*

$$|x_1 - y_1| + |x_2 - y_2| \leqslant |x_1 - x_2| + |y_1 - y_2| + 2\operatorname{diam}([x_1, y_1]^{+8\delta} \cap [x_2, y_2]^{+8\delta}).$$

2.2. Quasi-convexity

Let $\eta \geqslant 0$. A subset $Y \subset X$ is *η-quasi-convex* if every geodesic joining two points of Y is contained in $Y^{+\eta}$. For instance, geodesics are 2δ-quasi-convex. A subset $Y \subset X$ is *strongly quasi-convex* if it is 2δ-quasi-convex and for every $y, y' \in Y$, the induced path metric $|\cdot|_Y$ on Y satisfies

$$|y - y'|_X \leqslant |y - y'|_Y \leqslant |y - y'|_X + 8\delta.$$

Quasi-convexity in hyperbolic spaces has the following consequences.

Lemma 2.5 ([12, Chapitre 1, Proposition 3.1];[23, Lemma 2.4]). — *Let $\eta \geqslant 0$. Let $Y \subset X$ be an η-quasi-convex subset. Then for every $x \in X$ and for every y, $y' \in Y$,*

$$d(x, Y) \leqslant (y, y')_x + \eta + 3\delta.$$

Given a point $x \in X$ and a subset $Y \subset X$, then $y \in Y$ is a *projection* of x on Y if

$$|x - y| \leqslant d(x, Y) + \delta.$$

Lemma 2.6 ([12, Chapitre 2, Proposition 2.1];[14, Lemma 2.12]). — *Let $\eta \geqslant 0$. Let $Y \subset X$ be an η-quasi-convex subset.*

(i) *Let $x \in X$. Let y be a projection of x on Y. Then for every $y' \in Y$, $(x, y')_y \leqslant \eta + \delta$.*

(ii) *Let $i \in [\![1, 2]\!]$. Let $x_i \in X$. Let y_i be a projection of x_i on Y. Then,*

$$|y_1 - y_2| \leqslant \max\{|x_1 - x_2| - |x_1 - y_1| - |x_2 - y_2| + 2\varepsilon, \varepsilon\},$$

where $\varepsilon = 2\eta + 3\delta$.

Lemma 2.7 ([12, Chapitre 10, Proposition 1.2]; [14, Lemma 2.13]). — *Let $\eta \geqslant 0$. Let $Y \subset X$ be an η-quasi-convex subset. Then for every $\varepsilon \geqslant \eta$, the subset $Y^{+\varepsilon}$ is 2δ-quasi-convex.*

Lemma 2.8 ([22, Lemma 2.2.2 (2)]; [14, Lemma 2.16]). — *Let $i \in [\![1, 2]\!]$. Let $\eta_i \geqslant 0$. Let $Y_i \subset X$ be an η_i-quasi-convex subset. Then for every $\varepsilon \geqslant 0$,*

$$\operatorname{diam}(Y_1^{+\varepsilon} \cap Y_2^{+\varepsilon}) \leqslant \operatorname{diam}(Y_1^{+\eta_1+3\delta} \cap Y_2^{+\eta_2+3\delta}) + 2\varepsilon + 4\delta.$$

2.3. Isometries

Let G be a group acting by isometries on X. Let $x \in X$ be a point.

Classification of isometries. Recall that an isometry $g \in G$ is either *elliptic*, i.e. the orbit $\langle g \rangle \cdot x$ is bounded, *loxodromic*, i.e. the map $\mathbf{Z} \to X$ sending m to $g^m x$ is a quasi-isometric embedding or *parabolic*, i.e. it is neither loxodromic or elliptic, [12, Chapitre 9, Théorème 2.1]. Note that these definitions do not depend on the point x.

Translation lengths. To measure the action of an isometry $g \in G$ on X we define the *translation length* and the *stable translation length* as

$$\|g\| = \inf_{x \in X} |gx - x|, \quad \text{and} \quad \|g\|^\infty = \lim_{n \to +\infty} \frac{1}{n} |g^n x - x|.$$

Note that the definition of $\|g\|^\infty$ does not depend on the point x. These two lengths are related as follows, [12, Chapitre 10, Proposition 6.4].

$$\|g\|^\infty \leqslant \|g\| \leqslant \|g\|^\infty + 16\delta. \tag{2.1}$$

The isometry g is *loxodromic* if, and only if, its stable translation length is positive, [12, Ch. 10, Prop. 6.3].

Axis. The *axis* of $g \in G$ is the set

$$A_g = \{\, x \in X \,:\, |gx - x| \leqslant \|g\| + 8\delta \,\}.$$

Lemma 2.9 ([22, Proposition 2.3.3];[14, Proposition 2.28]). — *Let $g \in G$. Then A_g is 10δ-quasi-convex and $\langle g \rangle$-invariant. Moreover, for every $x \in X$,*

$$\|g\| + 2d(x, A_g) - 10\delta \leqslant |gx - x| \leqslant \|g\| + 2d(x, A_g) + 10\delta.$$

ℓ^∞-Energy. To measure the action of a finite subset of isometries $U \subset G$ on X we define the ℓ^∞-energy of U at x and the ℓ^∞-energy of U as

$$\mathrm{L}(U, x) = \max_{u \in U} |ux - x|, \quad \text{and} \quad \mathrm{L}(U) = \inf_{x \in X} \mathrm{L}(U, x).$$

The point x is *almost-minimizing the ℓ^∞-energy of U* if $\mathrm{L}(U, x) \leqslant \mathrm{L}(U) + \delta$. It is easy to see that the translation length and the ℓ^∞-energy are related as follows. For every $g \in U$,

$$\|g\| \leqslant \mathrm{L}(U). \tag{2.2}$$

2.4. Group action on a δ-hyperbolic space

Let G be a group acting by isometries on X.

Classification of group actions. We denote by ∂G the set of all accumulation points of an orbit $G \cdot x$ in the boundary ∂X. This set does not depend on the point x. One says that the action of G on X is

- *elliptic*, if ∂G is empty, or equivalently if one (hence any) orbit of G is bounded;
- *parabolic*, if ∂G contains exactly one point;
- *loxodromic*, if ∂G contains exactly two points;
- *non-elementary*, if ∂G contains at least 3 points, or equivalently if ∂G is infinite.

If the action of G is elliptic, parabolic or loxodromic, we will say that this action is *elementary*. In this context, being elliptic (respectively parabolic, loxodromic, etc) refers to the action of G on X. However, if there is no ambiguity we will simply say that G is elliptic (respectively parabolic, loxodromic, etc).

Lemma 2.10 ([15, Propositon 3.6]). — *If $|\partial G| \geqslant 2$, then G contains a loxodromic isometry.*

Acylindricity. For our purpose we require some properness for this action. We will use an acylindrical action on a metric space, keeping in mind the parameters that appear in the definition, [20, Proposition 5.31]. Recall that we assumed X to be δ-hyperbolic, with $\delta > 0$.

Definition 2.11 (Acylindrical action). — Let κ, $N > 0$. The group G acts (κ, N)-*acylindrically on the δ-hyperbolic space X* if the following holds: for every $x, y \in X$ with $|x - y| \geqslant \kappa$, the number of elements $u \in G$ satisfying $|ux - x| \leqslant 100\delta$ and $|uy - y| \leqslant 100\delta$ is bounded above by N.

Definition 2.12 (Global injectivity radius). — The *global injectivity radius* of the action of G on X is

$$\mathrm{T}(G, X) = \inf\{ \|g\|^{\infty} \ : \ g \in G \text{ } loxodromic \},$$

with the convention $\inf \varnothing = +\infty$.

Lemma 2.13 ([8, Lemma 4.2]; c.f. [18, Lemma 3.9]). — *Assume that the action of G on X is (κ, N)-acylindrical. Then*

$$\mathrm{T}(G, X) \geqslant \frac{\delta}{N}.$$

Loxodromic subgroups. Let $\kappa \geqslant 1$ and $l \geqslant 0$. Let $\gamma : [a, b] \to X$ be a rectifiable path with $a, b \in \mathbf{R} \cup \{-\infty, \infty\}$. We say that γ is a (κ, l)-*quasi-geodesic* if for all $[a', b'] \subset [a, b]$,

$$\text{length}(\gamma[a', b']) \leqslant k|\gamma(a') - \gamma(b')| + l.$$

Let $L \geqslant 0$. We say that γ is a L-*local* (κ, l)-*quasi-geodesic* if any subpath of γ whose length is at most L is a (κ, l)-quasi-geodesic. Let $H \leqslant G$ be a loxodromic subgroup with limit set $\partial H = \{\xi, \eta\}$. The H-*invariant cylinder*, denoted by C_H, is the open 20δ-neighborhood of all $10^3\delta$-local $(1, \delta)$-quasi-geodesics with endpoints ξ and η at infinity.

Lemma 2.14 (Invariant cylinder; [15, Lemma 3.13]). — *Let $H \leqslant G$ be a loxodromic subgroup. Then the subset C_H is invariant under the action of H and strongly quasiconvex.*

Lemma 2.15 ([14, Corollary 2.7]). — *Let $\gamma \colon I \to X$ be a $10^3\delta$-local $(1, \delta)$-quasi-geodesic. Then:*

(i) For every $t, t', s \in I$ such that $t \leqslant s \leqslant t'$, we have $(\gamma(t), \gamma(t'))_{\gamma(s)} \leqslant 6\delta$.

(ii) For every $x \in X$ and for every $y, y' \in \gamma$, we have $d(x, \gamma) \leqslant (y, y')_x + 9\delta$.

The *maximal loxodromic subgroup containing H* is the stabiliser of the set ∂H. For a loxodromic element $g \in G$, we denote by $E(g)$ the *maximal loxodromic subgroup containing g*. We define the *equivalence relation $\sim_g$* on G by $u \sim_g v$ if and only if $u^{-1}v \in E(g)$, for every $u, v \in G$. The *fellow travelling constant of a loxodromic element* $g \in G$ is

$$\Delta(g) = \sup\{ \operatorname{diam}(uA_g^{+20\delta} \cap vA_g^{+20\delta}) \ : \ u, v \in G, u \not\sim_g v \}.$$

Lemma 2.16 ([20, Proof of Proposition 6.29]). — *Assume that the action of G on X is (κ, N)-acylindrical. Let $g \in G$ be a loxodromic element. Then*

$$\Delta(g) \leqslant \kappa + (N + 2)\|g\|^{\infty} + 100\delta.$$

Lemma 2.17 ([20, Lemma 6.5]). — *Assume that the action of G on X is acylindrical. Let $g \in G$ be a loxodromic element. Then $E(g)$ is virtually cyclic.*

The subgroup $H^+ \leqslant G$ fixing pointwise ∂H is an at most index 2 subgroup of H. The next corollary is a well-known consequence of Lemma 2.10, Lemma 2.17 and [36, Lemma 4.1].

Corollary 2.18. — *Assume that the action of G on X is acylindrical. The set F of all elements of finite order of H^+ is a finite normal subgroup of H. Moreover there exists a loxodromic element $h \in H^+$ such that the map $F \rtimes_\phi \langle h \rangle \to H^+$ that sends (f, g) to fg is an isomorphism, where $\phi \colon \langle h \rangle \to \mathrm{Aut}(F)$ is the action by conjugacy of $\langle h \rangle$ on F.*

For a loxodromic element $g \in G$, we denote by $F(g)$ the set of all elements of finite order of $E^+(g)$. We say that g is *primitive* if its image in $E^+(g)/F(g)$ generates the quotient. The following lemma permits to produce primitive loxodromic elements uniformly. It will be useful during section section 4.

Lemma 2.19 ([31]; [4]; [25, Lemma 2.7]). — *For every $\kappa > 0$ and $N > 0$ there exists a positive integer n_0 with the following property. Let $U \subset G$ be a finite symmetric subset containing the identity. Assume that the action of G on X is (κ, N)-acylindrical. If $\mathrm{L}(U) > 50\delta$, then there exist a primitive loxodromic element $g \in U^{n_0}$ such that*

$$\|g\|^\infty \geqslant \frac{1}{2}\,\mathrm{L}(U).$$

Definition 2.20 (Loxodromic wideness). — The *loxodromic wideness* of the action of G on X is

$$\Phi(G, X) = \sup\{\, |F(g)| \;:\; g \in G \textit{ loxodromic} \,\},$$

with the convention $\sup \varnothing = -\infty$.

Lemma 2.21 ([34, Lem. 6.8]). — *Assume that the action of G on X is (κ, N)-acylindrical. Then*

$$\Phi(G, X) \leqslant N.$$

Classification of acylindrical actions. Following the proof of D. Osin [34, Theorem 1.1], one gets the following classification. It already appears in [28].

Lemma 2.22. — *Assume that the action of G on X is acylindrical. Then G satisfies exactly one of the following three conditions.*

(i) G is elliptic, or equivalently one (hence any) orbit of G is bounded.

(ii) G is loxodromic, or equivalently G is virtually cyclic and contains a loxodromic element.

(iii) G is non-elementary, or equivalently H contains a free group $\mathbf{F}_2$ of rank 2 and one (hence any) orbit of $\mathbf{F}_2$ is unbounded.

In particular, if the action of G on X is acylindrical, then every isometry $g \in G$ is either elliptic or loxodromic, [8]. The following trichotomy is a direct consequence of the previous lemma and [9, Theorem 13.1].

Lemma 2.23. — *Let G be a group acting acylindrically on a δ-hyperbolic space X. Let $U \subset G$ be a finite symmetric subset containing the identity. Then one of the following conditions holds:*

(T'1) $\mathrm{L}(U) \leqslant 10^4\delta$.

(T'2) The subgroup $\langle U\rangle$ is virtually cyclic and contains a loxodromic element.

(T'3) $\omega(U) \geqslant \frac{1}{10^3}\log 3$.

2.5. Small cancellation theory

Let G be a group acting by isometries on X. We recall that X is a δ-hyperbolic space.

Loxodromic moving family. The following definition generalises the conjugacy closure of a symmetrised set of relations in classical small cancellation theory.

Definition 2.24 (Loxodromic moving family). — A *loxodromic moving family* $\mathscr{Q}$ is a set of the form

$$\mathscr{Q} = \{\, (g\,\langle h\rangle\, g^{-1}, gC_h) \in \mathscr{Q} \,:\, g \in G,\, h \in \mathscr{L}\,\},$$

where $\mathscr{L} \subset G$ is a set of loxodromic elements and C_h stands for the $\langle h\rangle$-invariant cylinder.

Let $\mathscr{Q}$ be a loxodromic moving family. The *fellow travelling constant of* $\mathscr{Q}$ is

$$\Delta(\mathscr{Q}, X) = \sup\{\, \mathrm{diam}(Y_1^{+20\delta} \cap Y_2^{+20\delta}) \,:\, (H_1, Y_1) \neq (H_2, Y_2) \in \mathscr{Q}\,\}.$$

The *injectivity radius of* $\mathscr{Q}$ is

$$\mathrm{T}(\mathscr{Q}, X) = \inf\{\, \|h\| \,:\, h \in H - \{1\}, (H, Y) \in \mathscr{Q}\,\}.$$

Note that here we require the translation length and not the stable translation length, which was present in the definition of the global injectivity radius $\mathrm{T}(G, X)$. We denote $K = \langle\!\langle H \mid (H, Y) \in \mathscr{Q}\rangle\!\rangle$ and $\bar{G} = G/K$. We denote by $\pi\colon G \twoheadrightarrow \bar{G}$ the natural projection and write $\bar{g}$ for $\pi(g)$ for short, for every $g \in G$. The notation $\bar{U}$ may refer to either a subset of $\bar{G}$ or to $\pi(U)$, for some $U \subset G$.

Definition 2.25 (Small cancellation condition). — Let $\lambda > 0$ and $\varepsilon > 0$. We say that $\mathscr{Q}$ satisfies the *geometric $C''(\lambda, \varepsilon)$-small cancellation condition* if:

(SC1) $\Delta(\mathscr{Q}, X) < \lambda\, \mathrm{T}(\mathscr{Q}, X)$,

(SC2) $\mathrm{T}(\mathscr{Q}, X) > \varepsilon\delta$.

In that case we say that $\bar{G}$ is a *geometric $C''(\lambda, \varepsilon)$-small cancellation quotient.*

Cone-off space. Let $\rho > 0$. We denote by $\mathscr{Y}$ the collection of cylinders gC_h such that $g \in G$ and $h \in \mathscr{L}$. Let $Y \in \mathscr{Y}$. Note that $gC_h = C_{ghg^{-1}}$. The *cone of radius ρ over Y*, denoted by $Z_\rho(Y)$, is the quotient of $Y \times [0, \rho]$ by the equivalence relation that identifies all the points of the form $(y, 0)$. The *apex of the cone* $Z_\rho(Y)$ is the equivalence class of $(y, 0)$. By abuse of notation, we still write $(y, 0)$ for the equivalence class of $(y, 0)$. We denote by $\mathscr{V}$ the collection of apices of the cones over the elements of $\mathscr{Y}$. Let $\iota \colon Y \hookrightarrow Z_\rho(Y)$ be the map that sends y to (y, ρ). The *cone-off space of radius ρ over X relative to $\mathscr{Q}$*, denoted by $\dot{X}_\rho = \dot{X}_\rho(\mathscr{Q}, X)$, is the space obtained by attaching for every $Y \in \mathscr{Y}$, the cone $Z_\rho(Y)$ on X along Y according to $\iota \colon Y \hookrightarrow Z_\rho(Y)$. There is a natural metric on $\dot{X}_\rho(\mathscr{Q})$ and an action by isometries of G on $\dot{X}_\rho$.

Quotient space. The *quotient space of radius ρ over X relative to $\mathscr{Q}$*, denoted by $\bar{X}_\rho = \bar{X}_\rho(\mathscr{Q}, X)$, is the orbit space $\dot{X}_\rho/K$. We denote by $\zeta \colon \dot{X}_\rho \twoheadrightarrow \bar{X}_\rho$ the natural projection and write $\bar{x}$ for $\zeta(x)$ for short. Furthermore, we denote by $\bar{\mathscr{V}}$ the image in $\bar{X}_\rho$ of the apices $\mathscr{V}$. We consider $\bar{X}_\rho$ as a metric space equipped with the quotient metric, that is for every $x, x' \in \dot{X}_\rho$

$$|\bar{x} - \bar{x}'|_{\bar{X}} = \inf_{h \in K} |hx - x'|_{\dot{X}}.$$

We note that the action of G on $\dot{X}_\rho$ induces an action by isometries of $\bar{G}$ on $\bar{X}_\rho$.

Convention 2.26. — In what follows, we are going to assume that X is a metric graph whose edges all have the same constant length. This is to ensure that both the cone-off space $\dot{X}_\rho$ and the quotient space $\bar{X}_\rho$ are geodesic spaces, [10, I.7.19]. This is not a restrictive assumption, as explained in [20, Section 5.3].

The following lemma summarises Proposition 3.15 and Theorem 6.11 of [14]. It will be central in the proof of Theorem 1.2.

Lemma 2.27 (Small Cancellation Theorem [14]). — *There exist positive numbers δ_0, $\bar{\delta}$, Δ_0, ρ_0 satisfying the following. Let $0 < \delta \leqslant \delta_0$ and $\rho > \rho_0$. Let G be a group acting by isometries on a δ-hyperbolic space X. Let $\mathscr{Q}$ be a loxodromic moving family such that $\Delta(\mathscr{Q}, X) \leqslant \Delta_0$ and $\mathrm{T}(\mathscr{Q}, X) > 100\pi \sinh \rho$. Then:*

(i) $\bar{X}_\rho$ is a $\bar{\delta}$-hyperbolic space on which $\bar{G}$ acts by isometries.

(ii) Let $r \in (0, \rho/20]$. If for all $v \in \mathscr{V}$, the distance $|x - v| \geqslant 2r$ then the projection $\zeta \colon \dot{X}_\rho \to \bar{X}_\rho$ induces an isometry from $B(x, r)$ onto $B(\bar{x}, r)$.

(iii) Let $(H, Y) \in \mathscr{Q}$. If $v \in \mathscr{V}$ stands for the apex of the cone $Z_\rho(Y)$, then the natural projection $\pi \colon G \twoheadrightarrow \bar{G}$ induces an isomorphism from $\mathrm{Stab}(Y)/H$ onto $\mathrm{Stab}(\bar{v})$. □

Remark 2.28. — It is important to note that in this statement the constants δ_0, $\bar{\delta}$, Δ_0, ρ_0 are independent of G, X, $\mathscr{Q}$ or δ. Moreover δ_0 and Δ_0 (respectively ρ_0) can be chosen arbitrarily small (respectively large). We will refer to δ_0, $\bar{\delta}$, Δ_0, ρ_0 as *the constants of the Small Cancellation Theorem.*

For the remainder of this subsection, we choose δ, ρ, G, X, and $\mathcal{Q}$ satisfying the hypothesis of the Small Cancellation Theorem (Lemma 2.27). The following lemmas are consequence of the Small Cancellation Theorem.

Lemma 2.29 ([15, Proposition 5.16]). — *Let E be an elliptic (respectively loxodromic) subgroup of G for its action on X. Then the image of E through the natural projection $\pi\colon G \twoheadrightarrow \bar{G}$ is elliptic (respectively elementary) for its action on $\bar{X}_\rho$.*

Lemma 2.30 ([15, Proposition 5.17]). — *Let E be an elliptic subgroup of G for its action on X. Then the natural projection $\pi\colon G \twoheadrightarrow \bar{G}$ induces an isomorphism from E onto its image.*

Lemma 2.31 ([15, Proposition 5.18]). — *Let $\bar{E}$ be an elliptic subgroup of $\bar{G}$ for its action on $\bar{X}_\rho$. One of the following holds.*

(i) There exists an elliptic subgroup E of G for its action on X such that the natural projection $\pi\colon G \twoheadrightarrow \bar{G}$ induces an isomorphism from E onto $\bar{E}$.

(ii) There exists $\bar{v} \in \bar{\mathcal{V}}$ such that $\bar{E} \subset \mathrm{Stab}(\bar{v})$.

Lemma 2.32 ([19, Proposition 9.13]). — *Let $\bar{U} \subset \bar{G}$ be a finite set such that $\mathrm{L}(\bar{U}) \leqslant \rho/5$. If, for every $\bar{v} \in \bar{\mathcal{V}}$, the set $\bar{U}$ is not contained in $\mathrm{Stab}(\bar{v})$, then there exists a pre-image $U \subset G$ of $\bar{U}$ of energy $\mathrm{L}(U) \leqslant \pi \sinh \mathrm{L}(\bar{U})$.*

Lemma 2.33 (Greendlinger's Lemma). — *Let $x \in X$. Let $g \in G$. If $g \in K - \{1\}$, then there exists $(H, Y) \in \mathcal{Q}$ with the following property. Let y_0 an y_1 be the respective projections of x and gx on Y. Then*

$$|y_0 - y_1| > \mathrm{T}(H, X) - 2\pi \sinh \rho - 23\delta.$$

Remark 2.34. — The previous statement is obtained from [17, Theorem 3.5] after applying [17, Proposition 1.11], [14, Proposition 2.4 (2)] and [14, Lemma 2.31]. Note that in [17, Theorem 3.5] there is an extra assumption saying that the loxodromic moving family is finite up to conjugacy. That assumption is only needed to make sure that the action is co-compact, hence the quotient group hyperbolic. We don't need it here.

Lemma 2.35 ([20, Proposition 5.33]). — *If the action of G on X is acylindrical, then so is the action of $\bar{G}$ on $\bar{X}_\rho$.*

3. Reduced subsets

Let $\delta \geqslant 0$. In this section, we fix a group G acting by isometries on a δ-hyperbolic space X. The *set of the inverses in G of the elements of $U \subset G$* is represented by U^{-1}.

Definition 3.1. — Let $\alpha > 0$. We say that a finite subset $U \subset G$ is *α-reduced at $p \in X$* if $U \cap U^{-1} = \varnothing$ and for every pair of distinct $u_1, u_2 \in U \sqcup U^{-1}$,

$$(u_1 p, u_2 p)_p < \frac{1}{2} \min\{|u_1 p - p|, |u_2 p - p|\} - \alpha - 2\delta.$$

Remark 3.2. — If $U \subset G$ is α-reduced at $p \in X$, then $|up - p| > 2\alpha$, for every $u \in U \sqcup U^{-1}$.

We clarify some vocabulary. Let $U \subset G$ be a subset. A *letter* is an element of the *alphabet* $U \sqcup U^{-1}$. A *word over* $U \sqcup U^{-1}$ is any finite sequence $u_1 \cdots u_n$ with $u_i \in U \sqcup U^{-1}$. The number n is called the *length* of the the given word $u_1 \cdots u_n$. We denote by $|w|_U$ the length of any word w over $U \sqcup U^{-1}$. We admit the word of length 0, the *empty word.* We write $w_1 \equiv w_2$ to express letter-for-letter equality of words w_1 and w_2 over $U \sqcup U^{-1}$. A word $u_1 \cdots u_n$ over $U \sqcup U^{-1}$ is *reduced* if it does not contain a pair of adjacent letters of the form $u_i u_i^{-1}$ or $u_i^{-1} u_i$. The *free group* $\mathbf{F}(U)$ is the set of reduced words over $U \sqcup U^{-1}$ with the group operation “concatenate and reduce”. The *natural homomorphism* $\psi \colon \mathbf{F}(U) \to G$ is the evaluation of the elements of $\mathbf{F}(U)$ on G.

3.1. Broken geodesics

The next lemma is used to produce quasi-geodesics by concatenating some sequences of points of X with geodesics.

Lemma 3.3 (Broken Geodesic Lemma [4, Lemma 1]). **—** *Let $n \geqslant 2$. Let $x_0, \cdots, x_n$ be a sequence of $n+1$ points of X. Assume that*

$$(x_{i-1}, x_{i+1})_{x_i} + (x_i, x_{i+2})_{x_{i+1}} < |x_i - x_{i+1}| - 3\delta, \tag{3.1}$$

for every $i \in [\![1, n-2]\!]$. Then the following holds.

(i) $|x_0 - x_n| \geqslant \sum_{i=0}^{n-1} |x_i - x_{i+1}| - 2\sum_{i=1}^{n-1} (x_{i-1}, x_{i+1})_{x_i} - 2(n-2)\delta.$

(ii) $(x_0, x_n)_{x_j} \leqslant (x_{j-1}, x_{j+1})_{x_j} + 2\delta$, *for every $j \in [\![1, n-1]\!]$.*

(iii) The geodesic $[x_0, x_n]$ lies in the 5δ-neighbourhood of the broken geodesic $\gamma = [x_0, x_1] \cup \cdots \cup [x_{n-1}, x_n]$, while γ is contained in the r-neighbourhood of $[x_0, x_n]$, where

$$r = \sup_{1 \leqslant i \leqslant n-1} (x_{i-1}, x_{i+1})_{x_i} + 14\delta.$$

We verify the condition of Lemma 3.3 permitting to obtain broken geodesics.

Proposition 3.4. — *Let $\alpha > 0$. Let $U \subset G$ be an α-reduced subset at $p \in X$. Let $n \geqslant 2$. Let $w \equiv u_1 \cdots u_n$ be an element of $\mathbf{F}(U)$. Consider the sequence of $n+1$ points*

$$x_0 = p, \quad x_1 = u_1 p, \quad x_2 = u_1 u_2 p, \quad \cdots, \quad x_n = u_1 \cdots u_n p.$$

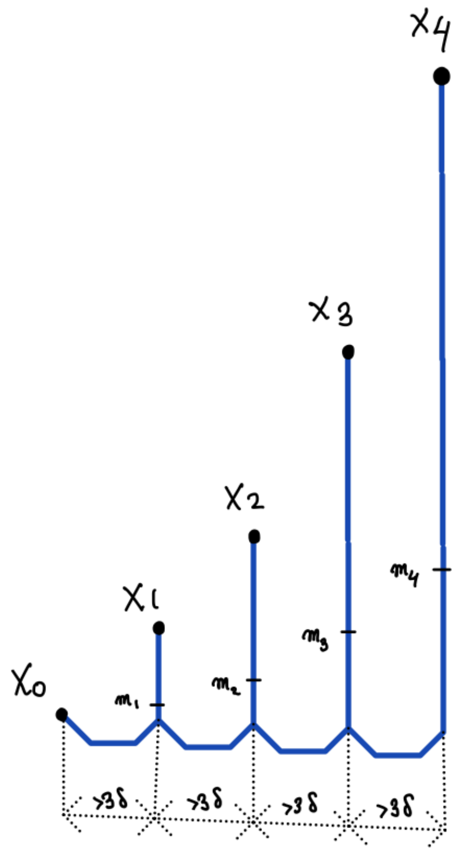

Figure 1: A sequence (x_i) satisfying Equation 3.1. This sequence does not correspond to a reduced word over a reduced subset since for every i, the midpoint m_i of the geodesic $[x_{i-1}, x_i]$ falls inside the overlap of two consecutive geodesics.

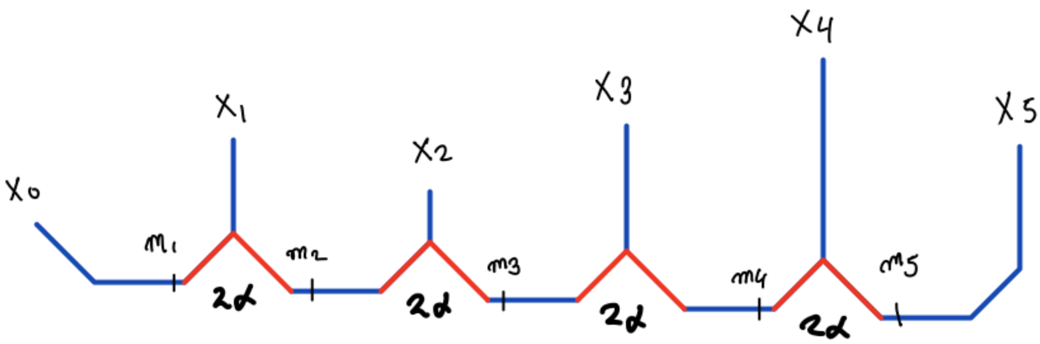

Figure 2: Another sequence (x_i) satisfying Equation 3.1. This sequence could correspond to a reduced word over an α-reduced subset since for every i, the midpoint m_i of the geodesic $[x_{i-1}, x_i]$ falls at distance at least α from the the overlap of two consecutive geodesics. The geodesic segments in red have length 2α. In particular, every geodesic $[x_{i-1}, x_i]$ that does not fall in any of the two extremes has length at least 2α.

Then

(i) $(x_{i-1}, x_{i+1})_{x_i} + (x_i, x_{i+2})_{x_{i+1}} < |x_i - x_{i+1}| - 2(\alpha + 2\delta)$, *for every* $i \in [\![1, n-2]\!]$.

(ii) $|wp - p| \geqslant \frac{1}{2}|u_1 p - p| + \frac{1}{2}|u_n p - p| + 2(n-1)(\alpha + \delta) + 2\delta$.

Proof. — (i) Let $i \in [\![1, n-2]\!]$. We have

$$(x_{i-1}, x_{i+1})_{x_i} = (u_i^{-1}p, u_{i+1}p)_p, \quad (x_i, x_{i+2})_{x_{i+1}} = (u_{i+1}^{-1}p, u_{i+2}p)_p$$

and $|x_i - x_{i+1}| = |p - u_{i+1}p|$. Since w is a reduced word over $U \sqcup U^{-1}$, we have $u_i^{-1} \neq u_{i+1}$ and $u_{i+1}^{-1} \neq u_{i+2}$. Hence we can apply the fact that the subset U is α-reduced at p, obtaining

$$(u_i^{-1}p, u_{i+1}p)_p < \frac{1}{2}|u_{i+1}p - p| - \alpha - 2\delta, \quad (u_{i+1}^{-1}p, u_{i+2}p)_p < \frac{1}{2}|u_{i+1}^{-1}p - p| - \alpha - 2\delta.$$

It remains to add the two above inequalities to obtain

$$(x_{i-1}, x_{i+1})_{x_i} + (x_i, x_{i+2})_{x_{i+1}} < |x_i - x_{i+1}| - 2(\alpha + 2\delta).$$

(ii) Since $n \geqslant 2$, applying (i) and Lemma 3.3 (i) to the sequence $x_0, \cdots, x_n$, we obtain

$$\begin{aligned} |wp - p| \geqslant{}& |u_1 p - p| + \sum_{i=2}^{n-1} |u_i p - p| + |u_n p - p| \\ & - (u_1^{-1}p, u_2 p)_p - \sum_{i=2}^{n-1} [(u_i^{-1}p, u_{i+1}p)_p + (u_{i-1}^{-1}p, u_i p)_p] - (u_{n-1}^{-1}p, u_n p) \\ & - 2(n-2)\delta. \end{aligned}$$

Since U is α-reduced at p,

$$\sum_{i=2}^{n-1} [(u_i^{-1}p, u_{i+1}p)_p + (u_{i-1}^{-1}p, u_i p)_p] < \sum_{i=2}^{n-1} |u_i p - p| - 2(n-2)(\alpha + 2\delta).$$

and

$$(u_1^{-1}p, u_2 p)_p < \frac{1}{2}|u_1 p - p| - \alpha - 2\delta, \quad (u_{n-1}^{-1}p, u_n p) < \frac{1}{2}|u_n p - p| - \alpha - 2\delta.$$

Consequently,

$$|wp - p| \geqslant \frac{1}{2}|u_1 p - p| + \frac{1}{2}|u_n p - p| + 2(n-1)(\alpha + \delta) + 2\delta.$$

□

3.2. Quasi-isometric embedding of a free group

Recall that $\mathrm{L}(U, p)$ denotes the ℓ^∞-energy of $U \subset G$ at $p \in X$ (subsection 2.3).

Proposition 3.5. — *Let $\alpha > 0$. Let $U \subset G$ be an α-reduced subset at $p \in X$. Then, for every $w \in \mathbf{F}(U)$, we have*

$$2\alpha|w|_U \leqslant |wp - p| \leqslant \mathrm{L}(U, p)|w|_U.$$

In particular, the natural homomorphism $\psi\colon \mathbf{F}(U) \to G$ is injective.

Proof. — Let $w \equiv u_1 \cdots u_n$ be an element of $\mathbf{F}(U)$. If $n = 0$, then there is nothing to do. If $n = 1$, then the result is a direct consequence of the fact that the subset U is α-reduced. Assume that $n \geqslant 2$. It follows from the triangle inequality that $|wp - p| \leqslant \mathrm{L}(U, p)n$. In regards to the second inequality, we apply Proposition 3.4 (ii) to the sequence of $n + 1$ points

$$x_0 = p, \quad x_1 = u_1, \quad x_2 = u_1u_2p, \quad \cdots, \quad x_n = wp = u_1 \cdots u_np,$$

to obtain

$$|wp - p| \geqslant \frac{1}{2}|u_1p - p| + \frac{1}{2}|u_np - p| + 2(n - 1)(\alpha + \delta) + 2\delta.$$

According to Remark 3.2, we have

$$\max\{|u_1p - p|, |u_np - p|\} \geqslant 2\alpha.$$

Hence,

$$|wp - p| \geqslant 2\alpha n.$$

Finally, if $w \in \mathbf{F}(U)$ is not the empty word, then $|wp - p| \geqslant 2\alpha$. By definition, $\alpha > 0$. Therefore $w \neq 1$ in G. Consequently, the natural homomorphism $\psi\colon \mathbf{F}(U) \to G$ is injective. □

3.3. Geodesic extension property

This is the main result of this section. Our proof is based on [19, Lemma 3.2].

Proposition 3.6. — *Let $\alpha > 0$. Let $U \subset G$ be an α-reduced subset at p. Let $w \equiv u_1 \cdots u_m$ and $w' \equiv u'_1 \cdots u'_{m'}$ be two elements of $\mathbf{F}(U)$. Then U satisfies the geodesic extension property, that is, if*

$$(p, w'p)_{wp} < \frac{1}{2}|u_mp - p| - \delta,$$

then w is a prefix of w'.

Remark 3.7. — The *geodesic extension property* has the following meaning: if the geodesic $[p, w'p]$ extends $[p, wp]$ as a path in X, then w' extends w as a word over $U \sqcup U^{-1}$.

Proof. — The proof is by contrapositive. Assume that w is not a prefix of w'. Let r be the largest integer such that $u_i = u'_i$, for every $i \in [\![1, r - 1]\!]$. In particular, $r \in [\![1, m]\!]$.

For simplicity, denote

$$q = u_1 \cdots u_{r-1} p = u'_1 \cdots u'_{r-1} p.$$

It follows from the four point inequality that

$$(p, w'p)_{wp} \geqslant \min\{(p,q)_{wp}, (q, wp')_{wp}\} - \delta. \tag{3.2}$$

From now on, the focus will be on showing that

$$\min\{(p,q)_{wp}, (q, wp')_{wp}\} \geqslant \frac{1}{2}|u_m p - p|.$$

Using the definition of Gromov product,

$$(p,q)_{wp} = |wp - q| - (p, wp)_q, \quad (q, w'p)_{wp} = |wp - q| - (wp, w'p)_q. \tag{3.3}$$

We are going to estimate $|wp - q|$, $(p, wp)_q$, and $(wp, w'p)_q$.

Claim 3.8. — $|wp - q| \geqslant \frac{1}{2}|u_r p - p| + \frac{1}{2}|u_m p - p| + 2(m-r)(\alpha+\delta)$.

Proof. — Note that $m - r + 1 \geqslant 1$. If $m - r + 1 = 1$, then there is nothing to do. If $m - r + 1 \geqslant 2$, then we apply Proposition 3.4 (ii) to the sequence of $m - r + 2$ points

$$q = u_1 \cdots u_{r-1} p, \quad u_1 \cdots u_r p, \quad u_1 \cdots u_{r+1} p, \quad \cdots, \quad wp = u_1 \cdots u_m p,$$

and we obtain

$$|wp - q| \geqslant \frac{1}{2}|u_r p - p| + \frac{1}{2}|u_m p - p| + 2(m-r)(\alpha+\delta).$$

□

For simplicity, denote

$$t = u_1 \cdots u_r p \quad \text{and} \quad t' = u'_1 \cdots u'_r p.$$

Claim 3.9. — $(p, wp)_q < \frac{1}{2}|u_r p - p|$.

Proof. — Applying Lemma 3.3 (ii) and Proposition 3.4 (i) to the sequence of $m+1$ points

$$p, \quad u_1 p, \quad u_1 u_2 p, \quad \cdots, \quad wp = u_1 \cdots u_m p,$$

we get

$$(p, wp)_q \leqslant (u_1 \cdots u_{r-2} p, t)_q + 2\delta.$$

Since U is α-reduced at p,

$$(u_1 \cdots u_{r-2} p, t)_q = (u_{r-1}^{-1} p, u_r p)_p < \frac{1}{2}|u_r p - p| - \alpha - 2\delta.$$

Consequently,

$$(p, wp)_q < \frac{1}{2}|u_r p - p| - \alpha.$$

This proves our claim. □

Claim 3.10. — $(wp, w'p)_q < \frac{1}{2}|u_r p - p|$.

Proof. — If $r-1 = m'$, then $w'p = q$ and the claim holds. Hence we can suppose that $r - 1 < m'$. It follows from the choice of r that $u_r \neq u'_r$. It follows from the four point inequality that

$$\min\{(t, wp)_q, (wp, w'p)_q, (w'p, t')_q\} \leqslant (t, t')_q + 2\delta.$$

Since U is α-reduced at p,

$$(t, t')_q = (u_r p, u'_r p)_q < \frac{1}{2}\min\{|u_r p - p|, |u'_r p - p|\} - \alpha - 2\delta.$$

Consequently,

$$\min\{(t, wp)_q, (wp, w'p)_q, (w'p, t')_q\} < \frac{1}{2}\min\{|u_r p - p|, |u'_r p - p|\} - \alpha. \tag{3.4}$$

We must prove that the minimum of Equation 3.4 is attained by $(wp, w'p)_q$. In order to do so, let's see first that the minimum of Equation 3.4 is not achieved by $(t, wp)_q$. Using the definition of Gromov product,

$$(t, wp)_q = |q - t| - (q, wp)_t.$$

By definition,

$$|q - t| = |u_r p - p|.$$

Recall that $m - r + 1 \geqslant 1$. If $m - r + 1 = 1$, we have

$$(q, wp)_t = (u_r^{-1} p, p)_p = 0.$$

If $m - r + 1 \geqslant 2$, applying Lemma 3.3 (ii) and Proposition 3.4 (i) to the sequence of $m - r + 2$ points

$$q = u_1 \cdots u_{r-1} p, \quad t = u_1 \cdots u_r p, \quad u_1 \cdots u_{r+1} p, \quad \cdots, \quad wp = u_1 \cdots u_m p,$$

we obtain

$$(q, wp)_t \leqslant (q, u_1 \cdots u_{r+1} p)_t + 2\delta.$$

Since U is α-reduced,

$$(q, u_1 \cdots u_{r+1} p)_t = (u_r^{-1} p, u_{r+1} p)_p < \frac{1}{2}|u_r p - p| - \alpha - 2\delta.$$

Consequently,

$$(t, wp)_q \geqslant \frac{1}{2}|u_r p - p| > \frac{1}{2}|u_r p - p| - \alpha.$$

Thus, the minimum of Equation 3.4 cannot be achieved by $(t, wp)_q$. Similarly, it cannot be achieved by $(w'p, t')_q$. Therefore, the only possibility is that it is achieved by $(wp, w'p)_q$. This proves our claim. □

Finally, combining Equation 3.2 and Equation 3.3 with our three claims, we obtain

$$(p, w'p)_{wp} \geqslant \min\{(p, q)_{wp}, (q, w'p)_{wp}\} - \delta > \frac{1}{2}|u_m p - p| - \delta.$$

□

4. Growth in groups acting on a δ-hyperbolic space

In this section, we review and adapt some of the techniques of M. Koubi. [31] – further developed by G. Arzhantseva and I. Lysenok, [4]. These techniques permit to study exponential growth rates of finite symmetric subsets in groups acting by isometries on hyperbolic spaces in the sense of M. Gromov. In particular, we clarify what are the involved parameters for acylindrical actions, which permits to obtain Theorem 4.8.

4.1. Growth of maximal loxodromic subgroups.

Let G be a group acting acylindrically on a hyperbolic space X. The goal of this subsection is to prove that the maximal loxodromic subgroups of G have some sort of uniform linear growth. We adapt an argument that was written for hyperbolic groups in [3, p. 484]. Recall that $\Phi(G, X)$ stands by the loxodromic wideness of the action of G on X (Definition 2.20). Given a loxodromic element $g \in G$, we denoted by $\|g\|^\infty$ its stable translation length (subsection 2.3) and by $E(g)$ the maximal loxodromic subgroup of G containing g (subsection 2.4).

Proposition 4.1. — *Let G be a group acting acylindrically on a hyperbolic space X. Let $U \subset G$ be a finite symmetric subset containing the identity. Let $g \in G$ be a primitive loxodromic element. Then, for every $n \geqslant 1$,*

$$|U^n \cap E(g)| \leqslant 2\Phi(G, X)\left(\frac{\mathrm{L}(U)}{\|g\|^\infty}4n + 1\right).$$

First, we focus on the case of the cyclic group generated by a loxodromic isometry.

Lemma 4.2. — *Let G be a group acting acylindrically on a hyperbolic space X. Let $U \subset G$ be a finite symmetric subset containing the identity. Let $g \in G$ be a loxodromic element. Then, for every $n \geqslant 1$,*

$$|U^n \cap \langle g \rangle| \leqslant \frac{\mathrm{L}(U)}{\|g\|^\infty}2n + 1.$$

Proof. — Let $n \geqslant 1$. We have,

$$|U^n \cap \langle g \rangle| = |\{ k \in \mathbf{Z} : g^k \in U^n \}|.$$

Since the subset U is symmetric,

$$|\{ k \in \mathbf{Z} : g^k \in U^n \}| \leqslant 2|\{ k \in \mathbf{N} - \{0\} : g^k \in U^n \}| + 1.$$

Let $k \geqslant 1$ such that $g^k \in U^n$. Since the element g is loxodromic, we have $\|g\|^\infty > 0$. Observe that

$$k = \frac{\|g^k\|^\infty}{\|g\|^\infty}.$$

Let $x \in X$. Then

$$\|g^k\|^\infty \leqslant \|g^k\| \leqslant |g^k x - x| \leqslant \max_{h \in U^n} |hx - x| = \mathrm{L}(U^n, x).$$

Since the point x is arbitrary, we get $\|g^k\|^\infty \leqslant \mathrm{L}(U^n)$. By the triangle inequality, $\mathrm{L}(U^n) \leqslant n\,\mathrm{L}(U)$. Hence,

$$k \leqslant \frac{\mathrm{L}(U)}{\|g\|^\infty} n.$$

Therefore,

$$|U^n \cap \langle g \rangle| \leqslant \frac{\mathrm{L}(U)}{\|g\|^\infty} 2n + 1.$$

□

We are ready for the proof of the proposition.

Proof of Proposition 4.1. — Let $F(g)$ be the set of all elements of finite order of $E^+(g)$. Recall that $F(g)$ is a normal subgroup of $E^+(g)$. Since the action of G on X is acylindrical and $E(g)$ is a loxodromic subgroup of G, there exists a loxodromic element $h \in E^+(g)$ such that the map

$$F(g) \rtimes_\phi \langle h \rangle \to E^+(g), (f, k) \mapsto fk$$

is a group isomorphism, where $\phi \colon \langle h \rangle \to \mathrm{Aut}(F(g))$ is the action by conjugacy of $\langle h \rangle$ on $F(g)$ (Corollary 2.18). Let $n \geqslant 1$. Let E_0 be a set of representatives of $E(g)/\langle h \rangle$. We have

$$|U^n \cap E(g)| = \sum_{r \in E_0} |U^n \cap r \langle h \rangle|.$$

First we are going to estimate $|E_0|$. By definition, $[E(g) \colon E^+(g)] \leqslant 2$. Since the homomorphism

$$\langle h \rangle \to F(g) \rtimes_\phi \langle h \rangle, k \mapsto (1, k)$$

is a split of the exact sequence,

$$0 \longrightarrow F(g) \xhookrightarrow{\iota} F(g) \rtimes_\phi \langle h \rangle \xrightarrow{\pi} \langle h \rangle \longrightarrow 0$$

we have $[E^+(g) \colon \langle h\rangle] = |F(g)| \leqslant \Phi(G,X)$. Consequently,

$$|E_0| \leqslant 2\Phi(G,X).$$

Since the action of G on X is acylindrical, we have $\Phi(G,X) < \infty$ (Lemma 2.21).

Now we are going to estimate $|U^n \cap r\langle h\rangle|$ for $r \in E_0$. We may assume that $U^n \cap r\langle h\rangle$ is non-empty. Then there exist $s \in U^n \cap r\langle h\rangle$. In particular $r\langle h\rangle = s\langle h\rangle$. Hence,

$$|U^n \cap r\,\langle h\rangle\,| = |U^n \cap s\,\langle h\rangle\,| = |s(s^{-1}U^n \cap \langle h\rangle)| = |s^{-1}U^n \cap \langle h\rangle\,|.$$

Since U is symmetric, $s^{-1} \in U^n$. Since U contains the identity, $s^{-1}U^n \subset U^{2n}$. Therefore,

$$|s^{-1}U^n \cap \langle h\rangle\,| \leqslant |U^{2n} \cap \langle h\rangle\,|.$$

According to Lemma 4.2,

$$|U^{2n} \cap \langle h\rangle\,| \leqslant \frac{\mathrm{L}(U)}{\|h\|^\infty}4n+1.$$

Consequently,

$$|U^n \cap r\,\langle h\rangle\,| \leqslant \frac{\mathrm{L}(U)}{\|h\|^\infty}4n+1.$$

Finally, since the element g is primitive, we have that $g \in \{h, h^{-1}\}$. It follows from our two estimations above that

$$|U^n \cap E(g)| \leqslant 2\Phi(G,X)\left(\frac{\mathrm{L}(U)}{\|g\|^\infty}4n+1\right).$$

□

Given a subset $U \subset G$ and a loxodromic element $g \in G$, we fix a set of representatives $U(g)$ of the equivalence relation induced on U by $\sim_g$. Recall that the equivalence relation $\sim_g$ on G was previously defined by $u \sim_g v$ if and only if $u^{-1}v \in E(g)$, for every $u, v \in G$ (subsection 2.4). The reason that makes the set $U(g)$ of interest is that the set of conjugates of g by the elements of $U(g)$ is a set of "independent" loxodromic elements and has the same size as $U(g)$. We obtain the following.

Corollary 4.3. — *Let G be a group acting acylindrically on a hyperbolic space X. Let $U \subset G$ be a finite symmetric subset containing the identity. Let $g \in G$ be a primitive loxodromic element. Let*

$$a_0 = 2\Phi(G,X)\left(\frac{\mathrm{L}(U)}{\|g\|^\infty}8+1\right).$$

Then,

$$|U(g)| \geqslant \frac{1}{a_0}|U|.$$

Proof. — Consider the surjective map $U \to U(g)$ that sends every element of U to its class representative in $U(g)$. We are going to estimate its injectivity. Let $u, v \in U$ such that $u \sim_g v$. By definition, $u^{-1}v \in E(g)$. Since the subset U is symmetric, $u^{-1}v \in U^2$. Therefore, $v \in u(U^2 \cap E(g))$. Note that $|u(U^2 \cap E(g))| = |U^2 \cap E(g)|$. Consequently, each $u \in U(g)$ has at most $|U^2 \cap E(g)|$ elements in its equivalence class. According to Proposition 4.1, $|U^2 \cap E(g)| \leqslant a_0$. Therefore,

$$|U(g)| \geqslant \frac{1}{a_0}|U|.$$

□

4.2. Producing reduced subsets

Recall that given a loxodromic element $g \in G$, we denoted by $\Delta(g)$ its fellow travelling constant (subsection 2.4). The goal of this subsection is to produce a reduced subset using the conjugates of a loxodromic isometry of large stable translation length. More precisely, we will prove the following.

Proposition 4.4. — *Let $\delta > 0$ and $\alpha > 0$. Let G be a group acting acylindrically on a δ-hyperbolic space X. Let $U \subset G$ be a finite symmetric subset containing a loxodromic element $g \in U$ such that $\|g\|^\infty > 10^3\delta$. Let $p \in X$. Let*

$$b_0 = \frac{200}{\|g\|^\infty}[\Delta(g) + \mathrm{L}(U,p) + \delta + \alpha].$$

Then for every $b \geqslant b_0$, the set $S = \{\, ug^bu^{-1} \;:\; u \in U(g) \,\}$ satisfies the following:

(i) $S \subset U^{b+2}$.

(ii) $|S| = |U(g)|$.

(iii) S *is α-reduced at p.*

Proof. — The conclusions (i) and (ii) are immediate. We are going to prove (iii) S is α-reduced at p (Definition 3.1). By construction, $S \cap S^{-1} = \varnothing$. Let $i \in [\![1,2]\!]$. Let $u_i \in U$. Let $\varepsilon_i \in \{-1,1\}$. Assume that the elements $u_1g^{\varepsilon_1 b}u_1^{-1}$ and $u_2g^{\varepsilon_2 b}u_2^{-1}$ are distinct.

Case $u_1 = u_2$. Since the elements $u_1g^{\varepsilon_1 b}u_1^{-1}$ and $u_2g^{\varepsilon_2 b}u_2^{-1}$ are distinct, we have $\varepsilon_1 = -\varepsilon_2$. Denote $h = u_1g^{\varepsilon_1 b}u_1^{-1}$. It is enough to prove that

$$(hp, h^{-1}p)_p \leqslant \frac{b}{2}\,\|g\|^\infty - \alpha - 2\delta.$$

Let η^- and η^+ be the points of ∂X fixed by $\langle h\rangle$ and $\gamma\colon \mathbf{R} \to X$ be an $\langle h\rangle$-invariant $10^3\delta$-local $(1,\delta)$-quasi-geodesic joining η^- to η^+. This choice is possible since $\|g\|^\infty > 10^3\delta$. It follows from Lemma 2.15 applied to γ that

$$(hp, h^{-1}p)_p \leqslant \mathrm{L}(U,p) + 6\delta.$$

It is clear that
$$\mathrm{L}(U,p) + 6\delta \leqslant \frac{b}{2}\,\|g\|^{\infty} - \alpha - 2\delta.$$

<u>*Case* $u_1 \neq u_2$</u>. In particular $u_1 \not\sim_g u_2$, which means that $u_1^{-1}u_2$ does not belong to $E(g)$.

Claim 4.5. — $d(p, A_g) \leqslant \frac{1}{2}\,\mathrm{L}(U,p) + 5\delta$.

Proof. — It follows from Lemma 2.9 that

$$d(p, A_g) \leqslant \frac{1}{2}|gp - p| + 5\delta.$$

Moreover, since $g \in U$, we have $|gp - p| \leqslant \mathrm{L}(U,p)$. This proves our claim. $\square$

Consider the points $x_i = u_i p$ and $y_i = u_i g^{\varepsilon_i b} p$.

Claim 4.6. — $\mathrm{diam}([x_1, y_1]^{+8\delta} \cap [x_2, y_2]^{+8\delta}) \leqslant \Delta(g) + \mathrm{L}(U,p) + 44\delta$.

Proof. — Denote $\sigma = d(p, A_g) + 10\delta$. We have,

$$\max\left\{d(x_i, u_i A_g), d(y_i, u_i A_g)\right\} \leqslant \sigma.$$

Recall that the axis A_g is 10δ-quasi-convex (Lemma 2.9). Hence, since $\sigma \geqslant 10\delta$, the subset $u_i A_g^{+\sigma}$ is 2δ-quasi-convex (Lemma 2.7). Consequently,

$$[x_i, y_i] \subset u_i A_g^{+\sigma+2\delta}.$$

Therefore,

$$\mathrm{diam}([x_1, y_1]^{+8\delta} \cap [x_2, y_2]^{+8\delta}) \leqslant \mathrm{diam}(u_1 A_g^{+\sigma+10\delta} \cap u_2 A_g^{+\sigma+10\delta}).$$

According to Lemma 2.8,

$$\mathrm{diam}(u_1 A_g^{+\sigma+10\delta} \cap u_2 A_g^{+\sigma+10\delta}) \leqslant \mathrm{diam}(u_1 A_g^{+13\delta} \cap u_2 A_g^{+13\delta}) + 2(\sigma + 10\delta) + 4\delta.$$

Moreover,
$$\mathrm{diam}(u_1 A_g^{+13\delta} \cap u_2 A_g^{+13\delta}) \leqslant \mathrm{diam}(u_1 A_g^{+20\delta} \cap u_2 A_g^{+20\delta}).$$

Since $u_1^{-1}u_2$ does not belong to $E(g)$,

$$\mathrm{diam}(u_1 A_g^{+20\delta} \cap u_2 A_g^{+20\delta}) \leqslant \Delta(g).$$

Since the action of G on X is acylindrical, we have $\Delta(g) < \infty$ (Lemma 2.16). Combining the above estimations with the previous claim, we obtain

$$\mathrm{diam}([x_1, y_1]^{+8\delta} \cap [x_2, y_2]^{+8\delta}) \leqslant \Delta(g) + \mathrm{L}(U,p) + 54\delta.$$

This proves our claim.

□

Denote $s_i = u_i g^{\varepsilon_i b} u_i^{-1}$.

Claim 4.7. — $(s_1p, s_2p)_p \leqslant \Delta(g) + 5\,\mathrm{L}(U,p) + 54\delta$.

Proof. — By definition,

$$(s_1p, s_2p)_p = \frac{1}{2}(|s_1p - p| + |s_2p - p| - |s_1p - s_2p|).$$

By the triangle inequality,

$$\begin{aligned} |s_ip - p| &\leqslant |x_i - y_i| + 2|u_ip - p|, \\ |s_1p - s_2p| &\geqslant |y_1 - y_2| - |u_1p - p| - |u_2p - p|. \end{aligned}$$

Consequently,

$$(s_1p, s_2p)_p \leqslant \frac{1}{2}(|x_1 - y_1| + |x_2 - y_2| - |y_1 - y_2|) + \frac{3}{2}(|u_1p - p| + |u_2p - p|).$$

Combining the previous claim with Lemma 2.4, we obtain

$$|x_1 - y_1| + |x_2 - y_2| - |y_1 - y_2| \leqslant |x_1 - x_2| + 2(\Delta(g) + \mathrm{L}(U,p) + 44\delta).$$

By the triangle inequality,

$$|x_1 - x_2| \leqslant |u_1p - p| + |u_2p - p|.$$

Moreover, since $u_i \in U$, we have $|u_ip - p| \leqslant \mathrm{L}(U,p)$. Combining the above estimations, we obtain

$$(s_1p, s_2p)_p \leqslant \Delta(g) + 5\,\mathrm{L}(U,p) + 44\delta.$$

This proves our claim.

□

Finally, note that

$$\frac{1}{2}\min\{|s_1p - p|, |s_2p - p|\} - \alpha - 2\delta \geqslant \frac{b}{2}\,\|g\|^\infty - \alpha - 2\delta.$$

Since $b \geqslant b_0$, we obtain

$$\frac{b}{2}\,\|g\|^\infty - \alpha - 2\delta > \Delta(g) + 5\,\mathrm{L}(U,p) + 54\delta.$$

Therefore, the previous claim implies that

$$(s_1p, s_2p)_p < \frac{1}{2}\min\{|s_1p - p|, |s_2p - p|\} - \alpha - 2\delta.$$

□

4.3. Growth trichotomy

We are going to combine the two previous subsections in the following result.

Theorem 4.8 (Theorem 1.10)**. —** *For every $\kappa > 0$ and $N > 0$, there exist an integer $c > 1$ with the following property. Let $\delta > 0$ and $\alpha > 0$. Let G be a group acting (κ, N)-acylindrically on a δ-hyperbolic space X. Let $U \subset G$ be a finite symmetric subset containing the identity. Let $p \in X$ be a point almost-minimizing the ℓ^∞-energy $\mathrm{L}(U)$. Then one of the following conditions holds:*

(T1) $\mathrm{L}(U) \leqslant 10^4 \max\{\kappa, \delta, \alpha\}$.

(T2) The subgroup $\langle U \rangle$ is virtually cyclic and contains a loxodromic element.

(T3) There exist a finite subset $S \subset G$ with the following properties:

- *(i)* $S \subset U^c$,
- *(ii)* $|S| \geqslant \max\left\{2, \frac{1}{c}|U|\right\}$,
- *(iii)* S *is α-reduced at p.*

Moreover, $\omega(U) \geqslant \frac{1}{c}\log|U|$.

Proof. — Let $\kappa > 0$ and $N > 0$. Let n_0 be the positive integer of Lemma 2.19 depending on κ and N. We fix auxiliar parameters

$$a_1 = 200Nn_0, \quad \text{and} \quad b_1 = 200(N+2) + 500n_0 + 700.$$

We put

$$c \geqslant \max\left\{a_1, n_0(b_1+2), \frac{2n_0(b_1+2)\log a_1}{\log 2}\right\}.$$

Let $\delta > 0$ and $\alpha > 0$. Let G be a group acting (κ, N)-acylindrically on a δ-hyperbolic space X. Let $U \subset G$ be a finite symmetric subset containing the identity. Let $p \in X$ be a point almost-minimizing the ℓ^∞-energy $\mathrm{L}(U)$. Assume that $\mathrm{L}(U) > 10^4 \max\{\kappa, \delta, \alpha\}$. Since $\mathrm{L}(U) > 50\delta$, according to Lemma 2.19 there exist a primitive loxodromic element $g \in U^{n_0}$ such that

$$\|g\|^\infty \geqslant \frac{1}{2}\mathrm{L}(U). \tag{4.1}$$

In particular $\|g\|^\infty \geqslant 10^3\delta$. Let $H = \langle U \rangle$. Note that the loxodromic g belongs to H. Assume in addition that the subgroup H is not virtually cyclic. We prove (T3). We are going to apply Corollary 4.3 and Proposition 4.4 to U^{n_0} and g. Let

$$a_0 = 2\Phi(G,X)\left(\frac{\mathrm{L}(U^{n_0})}{\|g\|^\infty}8 + 1\right), \quad b_0 = \frac{200}{\|g\|^\infty}[\Delta(g) + \mathrm{L}(U^{n_0}, p) + \delta + \alpha].$$

By the triangle inequality,

$$\mathrm{L}(U^{n_0}) \leqslant n_0\,\mathrm{L}(U), \quad \text{and} \quad \mathrm{L}(U^{n_0}, p) \leqslant n_0\,\mathrm{L}(U, p).$$

Since the point $p \in X$ is almost-minimizing the ℓ^∞-energy $\mathrm{L}(U)$, we have $\mathrm{L}(U, p) \leqslant \mathrm{L}(U) + \delta$. Since the action of G on X is (κ, N)-acylindrical, it follows from Lemma 2.21 and Lemma 2.16 that

$$\Phi(G, X) \leqslant N, \quad \text{and} \quad \Delta(g) \leqslant \kappa + (N+2)\,\|g\|^\infty + 100\delta.$$

Using the hypothesis $\mathrm{L}(U) > 10^4 \max\{\kappa, \delta, \alpha\}$ and Equation 4.1, we obtain,

$$\max\left\{\frac{\mathrm{L}(U)}{\|g\|^\infty}, \frac{\kappa}{\|g\|^\infty}, \frac{\delta}{\|g\|^\infty}, \frac{\alpha}{\|g\|^\infty}\right\} \leqslant 2.$$

Consequently, we obtain $a_0 \leqslant a_1$ and $b_0 \leqslant b_1$. Let $S = \{\, u g^{b_1} u^{-1} \;:\; u \in U^{n_0}(g)\,\}$.

The points (i) and (iii) follow from Proposition 4.4 (i) and (iii).

We are going to prove (ii). According to Proposition 4.4 (ii), we have $|S| = |U^{n_0}(g)|$. If $|U^{n_0}(g)| = 1$, then $u \sim_g g$, for every $u \in U^{n_0}$. Hence U^{n_0} is contained in $E(g)$. Since U contains the identity, $U \subset U^{n_0}$. Thus H is virtually cyclic (Lemma 2.17). Contradiction. Hence $|U^{n_0}(g)| \geqslant 2$. Further, it follows from Proposition 4.1 that $|U^{n_0}(g)| \geqslant \frac{1}{a_1}|U^{n_0}|$. Since U contains the identity, $|U^{n_0}| \geqslant |U|$. Therefore,

$$|S| \geqslant \max\left\{2, \frac{1}{a_1}|U|\right\}.$$

This implies our point (ii).

Let's verify the last conclusion about $\omega(U)$. Let $n \geqslant 1$. We have

$$|U^{n_0(b_1+2)n}| \geqslant |S^n| \geqslant |S|^n \geqslant \max\left\{2^n, \left(\frac{1}{a_1}|U|\right)^n\right\},$$

where the first inequality follows from (i); the second from (iii), which implies that the natural homomorphism $\mathbf{F}(S) \to G$ is injective (Proposition 5.16); and the third from (ii). Consequently,

$$\omega(U) = \limsup_{n\to\infty} \frac{1}{n_0(b_1+2)n} \log|U^{n_0(b_1+2)n}| \geqslant \frac{1}{n_0(b_1+2)} \max\left\{\log 2, \log\left(\frac{1}{a_1}|U|\right)\right\}.$$

Finally, note that

$$\frac{1}{a_1}|U| \geqslant |U|^{\frac{1}{2}} \Leftrightarrow \log|U| \geqslant 2\log a_1.$$

If $\log|U| \geqslant 2\log a_1$, we obtain

$$\omega(U) \geqslant \frac{1}{n_0(b_1+2)} \log\left(\frac{1}{a_1}|U|\right) \geqslant \frac{1}{2n_0(b_1+2)} \log|U|.$$

If $\log |U| < 2\log a_1$, we obtain

$$\omega(U) \geqslant \frac{1}{n_0(b_1+2)} \log 2 \geqslant \frac{\log 2}{2n_0(b_1+2)\log a_1} \log |U|.$$

□

5. Shortening and shortening-free words

In the context of classical small cancellation theory, *Greendlinger's Lemma* states that if a word over the free generating set of a free group represents the identity element in a small cancellation quotient, then it should contain a subword corresponding to a large portion of a relator. This section is structured as follows. First, we are going to formalise the notion of "large portion of a relator" with the definition of shortening word in the context of actions by isometries on hyperbolic spaces. Then, we are going to find a lower bound for the number of shortening-free words of free subgroups generated by reduced subsets of low energy. Finally, we will see that these shortening-free words embedd in geometric small cancellation quotients of appropriate parameters after using a suitable version of Greendlinger's Lemma (Lemma 2.33).

Global parameters and hypothesis for this section. Let δ_0 and Δ_0 be the constants of the Small Cancellation Theorem (Lemma 2.27). We fix once for all during this section

$$L_0 > 0, \quad \text{and} \quad \tau_0 = 10^6(\delta_0 + L_0 + \Delta_0).$$

Let

$$0 < \delta \leqslant \delta_0, \quad \alpha \geqslant 200\delta_0, \quad \text{and} \quad \tau \geqslant \tau_0.$$

Let G be a group acting by isometries on a δ-hyperbolic space X. Let $U \subset G$ be an α-reduced subset at $p \in X$ (Definition 3.1). Let $\mathscr{Q}$ be a loxodromic moving family (Definition 2.24). We assume that

$$0 < L(U,p) \leqslant L_0, \quad \text{and} \quad \Delta(\mathscr{Q}, X) \leqslant \Delta_0.$$

5.1. Shortening words

Here we study shortening words. Part of this subsection is based on [19, Section 3.1].

Definition 5.1 (*Shortening word*). — Let $w \equiv u_1 \cdots u_n$ be an element of $\mathbf{F}(U)$. Let $(H, Y) \in \mathscr{Q}$. We say that w is a *τ-shortening word over* (H,Y) if it satisfies the following. Consider the points $x_0 = p$ and $x_n = wp$. Let y_0 and y_n be respective projections of x_0 and x_n on Y. Then,

(S1) $|y_0 - y_n| > \tau$.

(S2) $|x_0 - y_0| < \frac{1}{2}|u_1 p - p| - 100\delta$, and $|x_n - y_n| < \frac{1}{2}|u_n p - p| - 100\delta$.

A *minimal τ-shortening word over* (H,Y) is a τ-shortening word over (H,Y) none of whose proper prefixes are τ-shortening words over (H,Y).

Remark 5.2. — Applying the triangle inequality, we observe that the choice $\tau \geqslant \tau_0$ implies that τ-shortening words over (H,Y) are distinct form the identity:

$$|x_0 - x_n| \geqslant |y_0 - y_n| - |x_0 - y_0| - |x_n - y_n| > 0.$$

Proposition 5.3. — *Let $w \equiv u_1 \cdots u_n$ be a τ-shortening word over $(H,Y) \in \mathcal{Q}$. Consider the sequence of $n+1$ points*

$$x_0 = p, \quad x_1 = u_1 p, \quad x_2 = u_1 u_2 p, \quad \cdots, \quad x_n = u_1 \cdots u_n p.$$

Let y_i be a projection of x_i on Y, for every $i \in [\![0,n]\!]$. Then,

$$|x_i - y_i| < \frac{1}{2} \min\{|u_i p - p|, |u_{i+1} p - p|\} - 100\delta,$$

for every $i \in [\![1,n-1]\!]$.

Proof. — Let $i \in [\![1,n-1]\!]$. Let z_i be a projection of x_i on $[y_0, y_n]$. Since Y is 10δ-quasi-convex (Lemma 2.14), there exist $z_i' \in Y$ such that $|z_i - z_i'| \leqslant 11\delta$. By definition,

$$|x_i - y_i| \leqslant d(x_i, Y) + \delta \leqslant |x_i - z_i'| + \delta.$$

By the triangle inequality,

$$|x_i - z_i'| \leqslant |x_i - z_i| + |z_i - z_i'|.$$

By definition, $|x_i - z_i| \leqslant d(x_i, [y_0, y_n]) + \delta$. According to Lemma 2.3,

$$d(x_i, [y_0, y_n]) \leqslant (y_0, y_n)_{x_i} + 4\delta.$$

We claim that $(y_0, y_n)_{x_i} \leqslant (x_0, x_n)_{x_i} + 2\delta$. It follows from the four point inequality that

$$\min\{(x_0, y_0)_{x_i}, (y_0, y_n)_{x_i}, (y_n, x_n)_{x_i}\} \leqslant (x_0, x_n)_{x_i} + 2\delta.$$

One can argue using the Broken Geodesic Lemma (Lemma 3.3) and the fact that w is a τ-shortening to prove that the minimum must be attained by $(y_0, y_n)_{x_i}$. Now applying the Broken Geodesic Lemma (Lemma 3.3 (ii)),

$$(x_0, x_n)_{x_i} \leqslant (x_{i-1}, x_{i+1})_{x_i} + 2\delta.$$

Moreover, $(x_{i-1}, x_{i+1})_{x_i} = (u_i^{-1} p, u_{i+1} p)_p$. Since the subset U is α-reduced and $\alpha \geqslant 200\delta$,

$$(u_i^{-1} p, u_{i+1} p)_p < \frac{1}{2} \min\{|u_i p - p|, |u_{i+1} p - p|\} - 118\delta.$$

Combining all the estimations, we obtain

$$|x_i - y_i| < \frac{1}{2}\min\{|u_i p - p|, |u_{i+1} p - p|\} - 100\delta.$$

□

Proposition 5.4. — *Let $w \equiv u_1 \cdots u_n$ be a τ-shortening word over $(H, Y) \in \mathcal{Q}$. The following holds.*

(i) We have

$$|w|_U \geqslant \frac{\tau - 50\delta}{\mathrm{L}(U, p)}.$$

(ii) If w is a minimal τ-shortening word over (H, Y), then

$$|w|_U \leqslant \frac{\tau}{\alpha} + 2.$$

Proof. — Consider the sequence of $n+1$ points

$$x_0 = p, \quad x_1 = u_1 p, \quad x_2 = u_1 u_2 p, \quad \cdots, \quad x_n = u_1 \cdots u_n p.$$

Let y_i be a projection of x_i on Y, for every $i \in [\![0, n]\!]$.

(i) Since $\mathrm{L}(U, p) > 0$ and w is distinct from the identity (Remark 5.2), it follows from the triangle inequality that,

$$|w|_U \geqslant \frac{|x_0 - x_n|}{\mathrm{L}(U, p)}.$$

According to (S1), we have $|y_0 - y_n| > \tau$. Since Y is 10δ-quasi-convex (Lemma 2.14) and $\tau \geqslant 23\delta$, the strong contraction property of Y (Lemma 2.6) implies

$$|x_0 - x_n| \geqslant |x_0 - y_0| + |y_0 - y_n| + |y_n - x_n| - 46\delta.$$

Consequently, $|x_0 - x_n| > \tau - 50\delta$. Therefore,

$$|w|_U \geqslant \frac{\tau - 50\delta}{\mathrm{L}(U, p)}.$$

(ii) Assume that w is a minimal τ-shortening word over (H, Y). Let $w' \equiv u_1 \cdots u_{n-1}$. By definition,

$$|w|_U = |w'|_U + 1.$$

In view of Proposition 3.4 (ii), we deduce

$$|w'p - p| \geqslant \frac{1}{2}|u_1 p - p| + \frac{1}{2}|u_{n-1} p - p| + \alpha(|w'|_U - 1).$$

By the triangle inequality,

$$|w'p - p| \leqslant |x_{n-1} - y_{n-1}| + |y_{n-1} - y_0| + |y_0 - x_0|.$$

Since w is a τ-shortening word over (H, Y), the property (S2) implies

$$|x_0 - y_0| < \frac{1}{2}|u_1 p - p| - 100\delta.$$

According to Proposition 5.3,

$$|x_{n-1} - y_{n-1}| < \frac{1}{2}|u_{n-1} p - p| - 100\delta.$$

Therefore, since w' is not a τ-shortening over (H, Y), we have $|y_{n-1} - y_0| \leqslant \tau$. Consequently, $|w'|_U \leqslant \frac{\tau}{\alpha} + 1$. Thus, $|w|_U \leqslant \frac{\tau}{\alpha} + 2$.

□

Proposition 5.5. — *Let (H_1, Y_1), $(H_2, Y_2) \in \mathcal{Q}$. Let $w \in \mathbf{F}(U)$. If w is a τ-shortening word over both (H_1, Y_1) and (H_2, Y_2), then $(H_1, Y_1) = (H_2, Y_2)$.*

Proof. — Assume that w is a τ-shortening word over (H_1, Y_1) and (H_2, Y_2). In order to prove that $(H_1, Y_1) = (H_2, Y_2)$, it is enough to show that $\operatorname{diam}(Y_1^{+20\delta} \cap Y_2^{+20\delta}) > \Delta(\mathcal{Q}, X)$. Since the subsets Y_1 and Y_2 are 10δ-quasi-convex (Lemma 2.14), it follows from Lemma 2.8 that

$$\operatorname{diam}(Y_1^{+20\delta} \cap Y_2^{+20\delta}) \geqslant \operatorname{diam}(Y_1^{+13\delta} \cap Y_2^{+13\delta}) \geqslant \operatorname{diam}(Y_1^{+2L_0} \cap Y_2^{+2L_0}) - 4L_0 - 4\delta_0.$$

Let $i \in [\![1, 2]\!]$. Let x_i and z_i be respective projections of p and wp on Y_i. We claim that $x_1, z_1 \in Y_1^{+2L_0} \cap Y_2^{+2L_0}$. Since w is a shortening word over (H_i, Y_i), it follows from (S2) that

$$\max\{|p - x_i|, |wp - z_i|\} \leqslant L_0.$$

According to the triangle inequality,

$$|x_1 - x_2| \leqslant |x_1 - p| + |p - x_2|, \quad |z_1 - z_2| \leqslant |z_1 - p| + |p - z_2|.$$

Consequently,

$$\max\{|x_1 - x_2|, |z_1 - z_2|\} \leqslant 2L_0.$$

Therefore, $x_1, z_1 \in Y_2^{+2L_0}$. This proves the claim. Thus,

$$\operatorname{diam}(Y_1^{+2L_0} \cap Y_2^{+2L_0}) \geqslant |x_1 - z_1|.$$

Since w is a shortening over (H_1, Y_1), it follows from (S1) that $|x_1 - z_1| > \tau$. Finally, since $\tau \geqslant \tau_0$, we obtain that $\operatorname{diam}(Y_1^{+20\delta} \cap Y_2^{+20\delta}) > \Delta(\mathcal{Q}, X)$. □

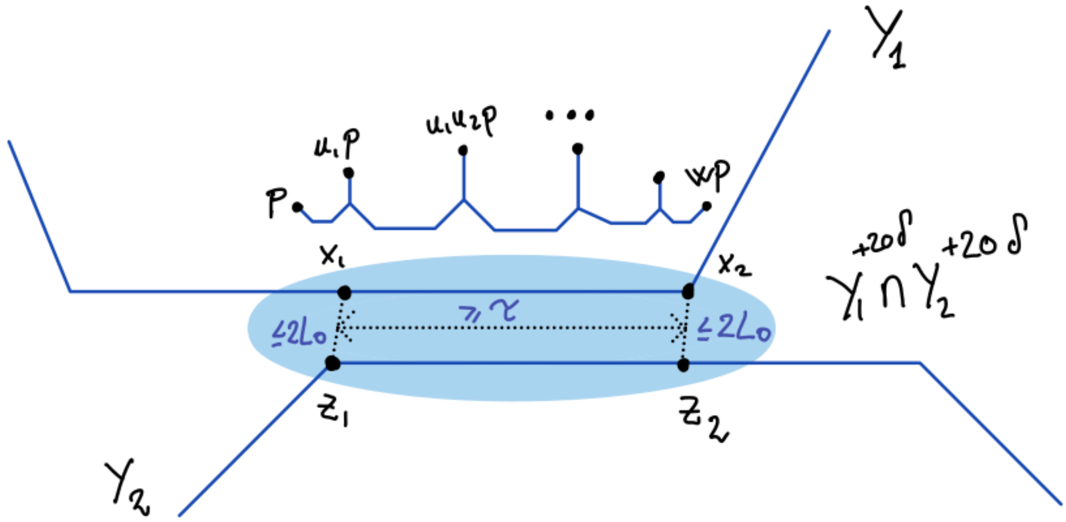

Figure 3: Scheme for the proof of Proposition 5.5.

Proposition 5.6. — *For every $(H,Y) \in \mathcal{Q}$, there exist at most two minimal τ-shortening words over (H,Y).*

Proof. — Let $(H,Y) \in \mathcal{Q}$. Let η^- and η^+ be the points of ∂X fixed by H and $\gamma\colon \mathbf{R} \to X$ be an $10^3\delta$-local $(1,\delta)$-quasi-geodesic joining η^- to η^+. Let q be a projection of p on γ. Without loss of generality, we may assume that $q = \gamma(0)$. Let $\mathscr{S}_{(H,Y)}$ denote the set of elements in $\mathbf{F}(U)$ that are τ-shortening words over (H,Y). Assume that $\mathscr{S}_{(H,Y)}$ is non-empty, otherwise the statement is true. We decompose $\mathscr{S}_{(H,Y)}$ in two sets as follows: an element $w \in \mathscr{S}_{(H,Y)}$ belongs to $\mathscr{S}^+_{(H,Y)}$ (respectively, $\mathscr{S}^-_{(H,Y)}$) if there is a projection $\gamma(t)$ of wp on γ with $t \geqslant 0$ (respectively, $t \leqslant 0$). Observe that a priori the sets $\mathscr{S}^-_{(H,Y)}$ and $\mathscr{S}^+_{(H,Y)}$ are not disjoint, but that will not be an issue for the rest of the proof.

Let $w_1, w_2 \in \mathscr{S}^+_{(H,Y)}$. Let $q_1 = \gamma(t_1)$ and $q_2 = \gamma(t_2)$ be the respective projections of w_1p and w_2p on γ. Without loss of generality, we may assume that $0 \leqslant t_1 \leqslant t_2$.

Claim 5.7. — The word w_1 is a prefix of w_2.

Proof. — We are going to apply the Geodesic Extension Property (Proposition 3.6). By the triangle inequality,

$$(p, w_2p)_{w_1p} \leqslant |w_1p - q_1| + (w_2p, p)_{q_1}. \tag{5.1}$$

Assume that $w_1 \equiv u_1 \cdots u_m$.

(a) Let's estimate $|w_1p - q_1|$. By definition, the H-invariant cylinder Y is contained in the 20δ-neighbourhood of γ. Consequently,

$$|w_1p - q_1| = d(w_1p, \gamma) \leqslant d(w_1p, Y) + 20\delta.$$

Since w_1 is a τ-shortening word over (H, Y), the property (S2) implies

$$d(w_1 p, Y) < \frac{1}{2}|u_m p - p| - 100\delta.$$

Therefore,

$$|w_1 p - q_1| < \frac{1}{2}|u_m p - p| - 80\delta. \tag{5.2}$$

(b) Let's estimate $(w_2 p, p)_{q_1}$. By definition,

$$(w_2 p, p)_{q_1} = \frac{1}{2}(|w_2 p - q_1| + |p - q_1| - |w_2 p - p|).$$

Since w_2 is a τ-shortening word over (H, Y), the property (S1) implies

$$|q_2 - q| > \tau.$$

Since Y is 10δ-quasi-convex (Lemma 2.14) and $\tau \geqslant 23\delta$, the strong contraction property of Y (Lemma 2.6) implies

$$|w_2 p - p| \geqslant |w_2 p - q_2| + |q_2 - q| + |q - p| - 46\delta.$$

Again by definition,

$$|q_2 - q| = |q_2 - q_1| + |q_1 - q| - 2(q_2, q)_{q_1}.$$

According to Lemma 2.15 (i),

$$(q_2, q)_{q_1} \leqslant 6\delta.$$

Note that here we have used the assumption $0 \leqslant t_1 \leqslant t_2$. By the triangle inequality,

$$|w_2 p - q_1| \leqslant |w_2 p - q_2| + |q_2 - q_1|.$$

Therefore,

$$|w_2 p - p| \geqslant |w_2 p - q_1| + |q_1 - p| - 58\delta.$$

Consequently,

$$(w_2 p, p)_{q_1} \leqslant 29\delta. \tag{5.3}$$

Finally, combining Equation 5.1, Equation 5.2 and Equation 5.3, we obtain

$$(p, w_2 p)_{w_1 p} \leqslant \frac{1}{2}|u_m p - p| - \delta.$$

Therefore, the Geodesic Extension Property (Proposition 3.6) implies that w_1 is a prefix of w_2. This proves our claim. $\square$

If w_1 is not a proper prefix of w_2, then the claim above implies that $w_1 = w_2$. Therefore $\mathscr{S}^+_{(H,Y)}$ has at most one element satisfying the statement of the proposition. By symmetry, $\mathscr{S}^-_{(H,Y)}$ has at most one element satisfying the statement. Therefore $\mathscr{S}_{(H,Y)}$ has at most two elements satisfying the statement.

$\square$

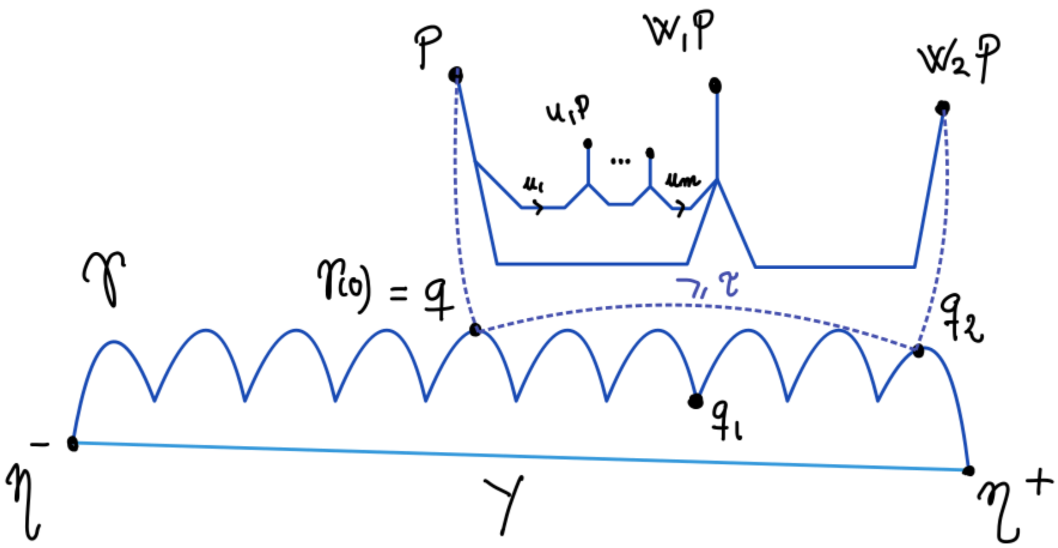

Figure 4: Scheme for the proof of Proposition 5.6.

5.2. The growth of shortening-free words

Here we count shortening-free words. The counting is based on [19, Section 3.22].

Definition 5.8 (*Shortening-free word*). — Let $w \equiv u_1 \cdots u_n$ be an element of $\mathbf{F}(U)$. Let $(H, Y) \in \mathscr{Q}$. We say that w *contains a τ-shortening word over* (H, Y) if w splits as $w \equiv w_0 w_1 w_2$, where w_1 is a τ-shortening word over (H, Y). We say that w is a *τ-shortening-free word* if for every $(H, Y) \in \mathscr{Q}$, the word w does not contain any τ-shortening word over (H, Y). We denote by $F(\tau) \subset \mathbf{F}(U)$ the subset of τ-shortening-free words.

Recall that the natural homomorphism $\mathbf{F}(U) \to G$ is injective (Proposition 3.5). Hence, we can safely identify the elements of $\mathbf{F}(U)$ with their images in G. The *ball* $B_U(n) \subset \mathbf{F}(U)$ of radius n is the set of reduced words over the alphabet $U \sqcup U^{-1}$ of length $|w|_U \leqslant n$, for every $n \geqslant 0$. Note that $B_U(n) = (U \sqcup U^{-1} \sqcup \{1\})^n$ when $n \geqslant 1$. Recall that we have fixed global hypothesis at the beginning of this section. The goal of this subsection is to obtain the following estimation.

Proposition 5.9. — *For every $\theta \in (0, 1/2)$, there exist $\tau_1 \geqslant \tau_0$ depending on θ, δ_0, L_0 and Δ_0 with the following property. If $|U| \geqslant 2$ and $\tau \geqslant \tau_1$, then for every $n \geqslant 0$, we have*

$$|F(\tau) \cap B_U(n+1)| \geqslant (1-\theta)(2|U|-1)|F(\tau) \cap B_U(n)|.$$

In particular, for every $n \geqslant 0$

$$|F(\tau) \cap B_U(n)| \geqslant (1-\theta)^n(2|U|-1)^n.$$

We are going to divide the proof of Proposition 5.9 into a few lemmas. First we fix some notations. We let

$$Z = \{ w \in \mathbf{F}(U) : w \equiv w_0 u, w_0 \in F(\tau), u \in U \sqcup U^{-1} \}.$$

For every $(H, Y) \in \mathscr{Q}$, we denote by $Z_{(H,Y)} \subset Z$ the set of elements $w \in Z$ that split as $w \equiv w_1 w_2$, where $w_1 \in F(\tau)$ and w_2 is a τ-shortening word over (H, Y).

Lemma 5.10. — *The set Z is contained in the disjoint union of $F(\tau)$ and $\bigcup_{(H,Y)\in\mathscr{Q}} Z_{(H,Y)}$.*

Proof. — The sets $F(\tau)$ and $\bigcup_{(H,Y)\in\mathscr{Q}} Z_{(H,Y)}$ are disjoint as a direct consequence of the definitions. Let $w \in Z - F(\tau)$. Since $w \in Z$, there exist $w_0 \in F(\tau)$ and $u \in U \sqcup U^{-1}$ such that $w \equiv w_0 u$. Since $w \notin F(\tau)$, there exist $(H, Y) \in \mathscr{Q}$ and a subword w_2 of w that is a τ-shortening word over (H, Y). It follows from the definition of $F(\tau)$ that every subword of w_0 must also be in $F(\tau)$. In particular, the word w_2 cannot be a subword of w_0. Hence, the only possibility is that w_2 is a suffix of w. Therefore, $w \in Z_{(H,Y)}$. ☐

Our Lemma 5.10 implies that for every $n \geqslant 0$,

$$|F(\tau) \cap B_U(n)| \geqslant |Z \cap B_U(n)| - \sum_{(H,Y)\in\mathscr{Q}} |Z_{(H,Y)} \cap B_U(n)|. \tag{5.4}$$

The next step is to estimate each term in the right side of the above inequality. The following lemma is a direct consequence of the definition of Z.

Lemma 5.11. — *For every $n \geqslant 0$,*

$$|Z \cap B_U(n+1)| = (2|U|-1)|F(\tau) \cap B_U(n)|.$$

Lemma 5.12. — *Let*

$$a = 2, \quad b = \left\lceil \frac{\tau_0}{200\delta_0} + 2 \right\rceil + 1, \quad M = \left\lfloor \frac{\tau - 50\delta_0}{L_0} \right\rfloor.$$

If $|U| \geqslant 2$, then for every $n \geqslant 0$,

$$\sum_{(H,Y)\in\mathscr{Q}} |Z_{(H,Y)} \cap B_U(n)| \leqslant a(2|U|-1)^b |F(\tau) \cap B_U(n-M)|.$$

Proof. — Assume that $|U| \geqslant 2$. Let $n \geqslant 0$. Note that for every $(H,Y) \in \mathcal{Q}$, the set $Z_{(H,Y)}$ is empty whenever there is no τ-shortening word over (H,Y). We denote by $\mathcal{Q}_0$ the set of $(H,Y) \in \mathcal{Q}$ for which there exist a τ-shortening word over (H,Y). We have,

$$\sum_{H \in \mathcal{Q}} |Z_{(H,Y)} \cap B_U(n)| = \sum_{(H,Y) \in \mathcal{Q}_0} |Z_{(H,Y)} \cap B_U(n)|.$$

The desired estimation is obtained from the two estimations of the claims below:

Claim 5.13. — $|Z_{(H,Y)} \cap B_U(n)| \leqslant a|F(\tau) \cap B_U(n-M)|$, for every $(H,Y) \in \mathcal{Q}_0$.

Proof. — Let $(H,Y) \in \mathcal{Q}_0$. Let $w \in Z_{(H,Y)} \cap B_U(n)$. Since $w \in Z_{(H,Y)}$, there exist $w_1 \in F(\tau)$ and a τ-shortening word w_2 over (H,Y) such that $w \equiv w_1 w_2$. We are going to describe the possible choices of w_1 and w_2. Since w is a reduced word over $U \sqcup U^{-1}$,

$$|w_1|_U = |w|_U - |w_2|_U.$$

According to Proposition 5.4 (i),

$$|w_2|_U \geqslant \frac{\tau - 50\delta_0}{L_0} \geqslant M \geqslant 0.$$

Therefore, $w_1 \in F(\tau) \cap B_U(n-M)$. Since $w \in Z$, the prefix consisting of all but the last letter is a τ-shortening free word. Thus, no proper prefix of w_2 is a τ-shortening word. It follows from Proposition 5.6 that there are most $a = 2$ possible choices for w_2. Therefore, there are at most $a|F(\tau) \cap B_U(n-M)|$ choices for w. This proves our claim. ☐

Claim 5.14. — $|\mathcal{Q}_0| \leqslant (2|U|-1)^b$

Proof. — Let $d = \left\lceil \frac{\tau_0}{200\delta_0} + 2 \right\rceil$. Since the free group $\mathbf{F}(U)$ has rank $|U| \geqslant 2$, we have

$$|B_U(d)| = \frac{|U|(2|U|-1)^d - 1}{|U|-1} \leqslant (2|U|-1)^{d+1} = (2|U|-1)^b.$$

Consequently, it suffices to show that there exists an injective map $\chi \colon \mathcal{Q}_0 \to B_U(d)$. Let $(H,Y) \in \mathcal{Q}_0$. By definition, there exist a τ-shortening word w over (H,Y). Note that since $\tau \geqslant \tau_0$, we have that w is a τ_0-shortening word over (H,Y). Let w' be the shortest prefix of w that is a τ_0-shortening word over (H,Y). In particular, w' is a minimal τ_0-shortening word over (H,Y). We define $\chi(H,Y) = w'$. Since $\alpha \geqslant 200\delta_0$, according to Proposition 5.4 (ii), $|w'|_U \leqslant d$. According to Proposition 5.5, there exist at most one $(H,Y) \in \mathcal{Q}$ such that w' is a τ_0-shortening word over (H,Y). Hence χ is well-defined and injective. This proves our claim. ☐

☐

Lemma 5.15. — *For every $\theta \in (0,1/2)$ and a, $b \geqslant 1$, there exist $M_0 \geqslant 0$ with the following* *property. Let*

$$\mu = (1-\theta)(2|U|-1), \quad \xi = a(2|U|-1)^b, \quad \text{and } \sigma = \frac{\theta}{2(1-\theta)\xi}.$$

If $|U| \geqslant 2$, then for every $M \geqslant M_0$, we have

$$\frac{1}{\mu^M} \leqslant \sigma.$$

Proof. — Let $\theta \in (0, 1/2)$ and $a, b \geqslant 1$. Let $M_0 = \max\left\{b, \frac{d_1}{d_2}\right\}$, where d_1, d_2 are constants depending only on θ, a, b whose exact value will be precised below. Let μ, ξ, σ as above. Assume that $|U| \geqslant 2$. Let $M \geqslant M_0$. In order to prove that $\frac{1}{\mu^M} \leqslant \sigma$, it is enough to show that $\log\left(\frac{1}{\sigma\mu^M}\right) \leqslant 0$. A first computation yields

$$\begin{aligned}
\log\left(\frac{1}{\sigma\mu^M}\right) &= -\log\sigma - \log\left(\mu^M\right), \\
\log(\sigma) &= \log\left(\frac{\theta}{2(1-\theta)a}\right) - b\log(2|U|-1), \\
\log\left(\mu^M\right) &= M\log(1-\theta) + M\log(2|U|-1).
\end{aligned}$$

Consequently,

$$\log\left(\frac{1}{\sigma\mu^M}\right) \leqslant (b-M)\log(2|U|-1) - M\log(1-\theta) - \log\left(\frac{\theta}{2(1-\theta)a}\right).$$

Since $M \geqslant b$ and $|U| \geqslant 2$, we have

$$(b-M)\log(2|U|-1) \leqslant (b-M)\log 3.$$

Therefore,

$$\log\left(\frac{1}{\sigma\mu^M}\right) \leqslant -M[\log 3 + \log(1-\theta)] + b\log 3 - \log\left(\frac{\theta}{2(1-\theta)a}\right).$$

We put

$$d_1 = b\log 3 + \log(2a) - \log\left(\frac{\theta}{1-\theta}\right), \quad d_2 = \log 3 + \log(1-\theta).$$

Since $a \geqslant 1$, $b \geqslant 1$ and $\theta \in (0, 1/2)$, we have $\min\{d_1, d_2\} > 0$. Finally, since $M \geqslant \frac{d_1}{d_2}$, we obtain, $\log\left(\frac{1}{\sigma\mu^M}\right) \leqslant 0$. □

We are ready to prove the proposition.

Proof of Proposition 5.9. — Let $\theta \in (0, 1/2)$. We are going to define the constant τ_1. Let

$$a = 2, \quad b = \left\lceil \frac{\tau_0}{200\delta_0} + 2 \right\rceil + 1.$$

Let $M_0 \geqslant 0$ be the constant of Lemma 5.15 depending on θ, a, b. We put

$$\tau_1 = \max\{\tau_0, L_0(M_0+1) + 50\delta_0\}.$$

Assume that $|U| \geqslant 2$ and $\tau \geqslant \tau_1$. We define the auxiliary parameters

$$\mu = (1-\theta)(2|U|-1), \quad \xi = a(2|U|-1)^b, \quad \sigma = \frac{\theta}{2\xi(1-\theta)}, \quad \text{and } M = \left\lfloor \frac{\tau - 50\delta_0}{L_0} \right\rfloor.$$

In particular, $M \geqslant M_0$. For every $n \geqslant 0$, we let

$$c(n) = |F(\tau) \cap B_U(n)|.$$

We must prove that for every $n \geqslant 1$,

$$c(n) \geqslant \mu c(n-1).$$

The proof goes by induction on n:

Base step. We claim that $c(1) \geqslant \mu$. Note that $B_U(1) = U \sqcup U^{-1} \sqcup \{1\}$. Therefore, it is enough to show that $U \sqcup U^{-1} \sqcup \{1\}$ is contained in $F(\tau)$. Let $w \in U \sqcup U^{-1} \sqcup \{1\}$. In particular, $|w|_U = 1$. Therefore, $w \in F(\tau)$ if and only if for every $(H,Y) \in \mathcal{Q}$, the element w is not a τ-shortening word over (H,Y). According to Proposition 5.4 (i), for every $(H,Y) \in \mathcal{Q}$ and for every τ-shortening word v over (H,Y), we have $|v|_U \geqslant \frac{\tau - 50\delta_0}{L_0}$. Since $\tau \geqslant \tau_0$, we have $1 < \frac{\tau-50\delta_0}{L_0}$. Consequently, $w \in F(\tau)$. This proves our claim.

Inductive step. Let $n \geqslant 1$. Assume that $c(m) \geqslant \mu c(m-1)$, for every $m \in [\![1,n]\!]$. We claim that $c(n+1) \geqslant \mu c(n)$. According to Equation 5.4,

$$c(n+1) \geqslant |Z \cap B_U(n+1)| - \sum_{(H,Y)\in\mathcal{Q}} |Z_{(H,Y)} \cap B_U(n+1)|.$$

It follows from Lemma 5.11 and Lemma 5.12 that

$$c(n+1) \geqslant (2|U|-1)c(n) - \xi c(n+1-M).$$

The induction hypothesis implies that for every $k \geqslant 0$, we have $c(n-k) \leqslant \mu^{-k} c(n)$. Note that $M-1 \geqslant 0$. Therefore, specifying the choice $k = M-1$, we obtain

$$c(n+1) \geqslant \left(1 - \frac{\xi\mu}{2|U|-1}\frac{1}{\mu^M}\right)(2|U|-1)c(n).$$

Recall that we defined $\mu = (1-\theta)(2|U|-1)$. Hence, in order to prove our claim, it is enough to show that

$$\frac{\xi\mu}{2|U|-1}\frac{1}{\mu^M} \leqslant \theta.$$

Since $M \geqslant M_0$, it follows from Lemma 5.15 that

$$\frac{1}{\mu^M} \leqslant \sigma.$$

Finally, note that

$$\frac{\xi\mu}{2|U|-1}\sigma = \frac{\xi(1-\theta)(2|U|-1)}{2|U|-1}\frac{\theta}{2\xi(1-\theta)} = \frac{\theta}{2} \leqslant \theta.$$

This proves our claim. □

5.3. The injection of shortening-free words

Let ρ_0 be the constant of the Small Cancellation Theorem (Lemma 2.27). Let $\tau_1 \geqslant \tau_0$ be the constant of Proposition 5.9 depending on $\theta = 1/3$, δ_0, L_0 and Δ_0. Let

$$\rho \geqslant \max\{\rho_0, \log(2[4\tau_1 + 23\delta_0] + 1)\}.$$

In addition to the global hypothesis for this section, we assume that

$$\mathrm{T}(\mathscr{Q}, X) \geqslant 100\pi \sinh \rho.$$

Denote $K = \langle\langle H \mid (H,Y) \in \mathscr{Q} \rangle\rangle$ and $\bar{G} = G/K$. The goal of this subsection is to prove:

Proposition 5.16. — *There exists $\tau_2 \geqslant \tau_1$ depending on δ_0, L_0 and Δ_0 with the following property. The restriction of the natural homomorphism $\mathbf{F}(U) \to \overline{G}$ to the subset of τ_2-shortening-free words is an injection.*

Lemma 5.17. — *Let $w \equiv u_1 \cdots u_n$ be an element of $\mathbf{F}(U)$. Let $(H,Y) \in \mathscr{Q}$. Let y_0 and y_n be respective projections of p and wp on Y. If $|y_0 - y_n| > 2\tau$, then w contains a $(2\tau - \tau_0)$-shortening word over a conjugate of (H,Y).*

Proof. — Consider the sequence of $n+1$ points

$$x_0 = p, \quad x_1 = u_1 p, \quad x_2 = u_1 u_2 p, \quad \cdots, \quad x_n = u_1 \cdots u_n p.$$

Let y_i be a projection of x_i on Y, for every $i \in [\![0, n]\!]$. Assume that $|y_0 - y_n| > 2\tau$. Since Y is 10δ-quasi-convex (Lemma 2.14) and $\tau \geqslant 23\delta$, the strong contraction property of Y (Lemma 2.6) implies that there exist $y_0', y_n' \in [p, wp]$ such that

$$\max\{|y_0 - y_0'|, |y_n - y_n'|\} \leqslant 23\delta \leqslant 23\delta_0.$$

Consider the broken geodesic

$$\gamma_w = \bigcup_{i=1}^{n} (u_1 \cdots u_{i-1})[p, u_i p].$$

Let y_0'' and y_n'' be respective projections of y_0' and y_n' on γ_w. Up to permuting y_0' and y_n' we may assume that p, y_0'', y_n'' and wp are ordered in this way along γ_w. In particular, there are $i \leqslant n-1$ and $j \leqslant n-1$ such that $y_0'' \in (u_1 \cdots u_i)[p, u_{i+1}p]$ and $y_n'' \in (u_1 \cdots u_j)[p, u_{j+1}p]$. Since y_0'' comes before y_n'' on γ_w, we have $i \leqslant j$. Let $w_0 \equiv u_1 \cdots u_{i+1}$ and take the word w_1 such that $w_0 w_1 \equiv u_1 \cdots u_j$. We are going to prove that w_1 is a $(2\tau - \tau_0)$-shortening word over $(w_0^{-1} H w_0, w_0^{-1} Y)$. The property (S2) follows from the fact that U is $200\delta_0$-reduced at p and from the Broken Geodesic Lemma (Lemma 3.3). Let's prove (S1), i.e. $|y_{i+1} - y_j| > 2\tau - \tau_0$. By the triangle inequality,

$$\begin{aligned}
|y_{i+1} - y_j| &\geqslant |y_0 - y_n| - |y_0 - y_{i+1}| - |y_n - y_j|,\\
|y_0 - y_{i+1}| &\leqslant |y_0 - y_0'| + |y_0' - y_0''| + |y_0'' - x_{i+1}| + |x_{i+1} - y_{i+1}|,\\
|y_n - y_j| &\leqslant |y_n - y_n'| + |y_n' - y_n''| + |y_n'' - x_j| + |x_j - y_j|.
\end{aligned}$$

Since $[x_0, x_n]$ is contained in the 5δ-neighbouhood of γ_w (Lemma 3.3 (iii)),

$$\max\{|y_0' - y_0''|, |y_n' - y_n''|\} \leqslant 5\delta \leqslant 5\delta_0.$$

Since $y_0'' \in (u_1 \cdots u_i)[p, u_{i+1}p]$ and $y_n'' \in (u_1 \cdots u_j)[p, u_{j+1}p]$,

$$\max\{|y_0'' - x_{i+1}|, |y_n'' - x_j|\} \leqslant \mathrm{L}(U, p) \leqslant L_0.$$

It follows from (S2) that,

$$\max\{|x_{i+1} - y_{i+1}|, |x_j - y_j|\} \leqslant \mathrm{L}(U, p) \leqslant L_0.$$

Combining the previous estimations, we obtain $|y_{i+1} - y_j| > 2\tau - \tau_0$. Note that $2\tau - \tau_0 \geqslant \tau_0$. □

Proof of Proposition 5.16. — We put $\tau_2 = 2\tau_1 - \tau_0$. Let $w_1, w_2 \in \mathbf{F}(U)$ be two τ_2-shortening-free words such that $w_1 w_2 \in K$. Assume for a contradiction that $w_1 w_2$ is not the identity as an element of G. According to Greendlinger's Lemma (Lemma 2.33), there exist $(H, Y) \in \mathscr{Q}$ such that if y_0 and y_2 are respective projections of p and $w_1 w_2 p$ on Y, then

$$|y_0 - y_2| > T(H, X) - 2\pi \sinh \rho - 23\delta.$$

By definition, $T(H, X) \geqslant T(\mathscr{Q}, X)$. By hypothesis

$$T(\mathscr{Q}, X) \geqslant 100\pi \sinh \rho, \quad \text{and} \quad \delta \leqslant \delta_0.$$

Therefore,

$$|y_0 - y_2| > \frac{e^\rho - 1}{2} - 23\delta_0.$$

The choice of ρ now implies that

$$|y_0 - y_2| > 4\tau_1$$

Let y_1 be a projection of $w_1 p$ on Y. Note that $w_1^{-1} y_1$ and $w_1^{-1} y_2$ are respective projections of p and $w_2 p$ on $w_1^{-1} Y$. Also, $(w_1^{-1} H w_1, w_1^{-1} Y) \in \mathcal{Q}$. Since w_1 and w_2 are τ_2-shortening-free words, it follows from Lemma 5.17 that

$$\max \{|y_0 - y_1|, |y_1 - y_2|\} < 2\tau_1.$$

By the triangle inequality,

$$|y_0 - y_2| \leqslant |y_0 - y_1| + |y_1 - y_2| \leqslant 4\tau_1.$$

Contradiction. Hence $w_1 w_2 = 1$. □

6. Growth in small cancellation groups

The goal of this section is to prove Theorem 1.2. We start with the following lemma.

Lemma 6.1. — *Let $a \geqslant 0$, $b \geqslant a$. Let G be a group acting acylindrically on a δ-hyperbolic space X. Let $U \subset G$ be a finite symmetric subset containing the identity such that $\mathrm{L}(U) \leqslant b$. Let $\Gamma = \langle U \rangle$. One of the following holds.*

(i) Γ is elliptic.

(ii) There exist $n \geqslant 1$ depending on U such that

$$a < \mathrm{L}(U^n) \leqslant 2b.$$

Proof. — Assume that Γ is not elliptic. Since the action of G on X is acylindrical, there exists a loxodromic element $g \in \Gamma$ (Lemma 2.22).

Claim 6.2. — There exists $M_0 \geqslant 1$ depending on U such that for every $M \geqslant M_0$,

$$\mathrm{L}(U^M) > a.$$

Proof. — According to Lemma 2.13, the global injectivity radius $\mathrm{T}(G, X)$ is distinct from zero. Let $m \geqslant \frac{a+\delta}{\mathrm{T}(G,X)}$. Since $g \in \Gamma$ and U is a symmetric generating set, there exists $M_0 \geqslant 1$ depending on U such that $g^m \in U^{M_0}$. Let $M \geqslant M_0$. Let $p \in X$ almost-minimizing the ℓ^∞-energy $\mathrm{L}(U^{M_0})$. We have,

$$\mathrm{L}(U^M, p) \geqslant \mathrm{L}(U^{M_0}, p) \geqslant |g^m p - p| \geqslant \|g^m\|^\infty = m\|g\|^\infty > a + \delta.$$

Hence $\mathrm{L}(U^M) > a$. This proves our claim. □

It follows from the claim above that there exist a smallest number $n \geqslant 1$ depending on U such that $\mathrm{L}(U^n) > a$. If $n = 1$, then we have $\mathrm{L}(U) \leqslant b$ by hypothesis. Therefore, $\mathrm{L}(U) \leqslant 2b$. If $n \geqslant 2$, then $n \leqslant 2(n-1)$. Since U contains the identity, $U^n \subset U^{2(n-1)}$. By the triangle inequality,

$$\mathrm{L}(U^n) \leqslant \mathrm{L}(U^{2(n-1)}) \leqslant 2\,\mathrm{L}(U^{n-1}) \leqslant 2a \leqslant 2b.$$

□

Hypothesis for the remainder of this section. Recall that the constants of the Small Cancellation Theorem (Lemma 2.27) are δ_0, $\bar{\delta}$, Δ_0, ρ_0. We can choose δ_0 arbitrarily small (Remark 2.28). For convenience, we will assume

$$\delta_0 \leqslant \frac{\pi \sinh 10^4 \bar{\delta}}{10^4 \cdot 200}.$$

We define the first geometric small cancellation parameter:

$$\lambda \leqslant \frac{\Delta_0}{100\pi \sinh \rho_0}.$$

Let $N > 0$. Let $c > 1$ be the constant of Theorem 4.8 depending only on the acylindricity parameters (δ_0, N). We fix an auxiliar parameter that will be used to bound the ℓ^∞-energy:

$$L_0 = c \cdot (2\pi \sinh 10^4 \bar{\delta} + \delta_0).$$

Let τ_1 and τ_2 be the constants of Proposition 5.16 depending on δ_0, L_0 and Δ_0. Let

$$\rho \geqslant \max\left\{\rho_0, \log(2[4\tau_1 + 23\delta_0] + 1), 5 \cdot 10^4 \bar{\delta}\right\}.$$

Let $\delta > 0$ and $\kappa \geqslant \delta$. We define the second geometric small cancellation parameter:

$$\varepsilon \geqslant \frac{100\pi \sinh \rho}{\delta_0} \cdot \frac{\kappa}{\delta}.$$

Let G be a group acting (κ, N)-acylindrically on a δ-hyperbolic space X. Let $\mathscr{Q}$ be a loxodromic moving family satisfying the geometric $C''(\lambda, \varepsilon)$-small cancellation condition for the action of G on X. We define a rescaling parameter

$$\sigma = \min\left\{\frac{\delta_0}{\kappa}, \frac{\Delta_0}{\Delta(\mathscr{Q}, X)}\right\}.$$

Remark 6.3. — Instead of working with the action of G on X, we will work with the action of G on the rescaled space $\mathcal{X}$.

The space $\mathcal{X}$ is $\sigma\delta$-hyperbolic and the action of G on $\mathcal{X}$ is $(\sigma\kappa, N)$-acylindrical. Note that

$$\sigma\delta \leqslant \sigma\kappa \leqslant \delta_0,$$

where the first inequality comes from the hypothesis $\kappa \geqslant \delta$. In particular, the action of G on $\mathcal{X}$ is (δ_0, N)-acylindrical for the hyperbolicity constant $\sigma\delta$. Besides, we have

$$\Delta(\mathscr{Q}, \mathcal{X}) \leqslant \sigma\Delta(\mathscr{Q}, X) \leqslant \Delta_0,$$
$$\mathrm{T}(\mathscr{Q}, \mathcal{X}) \geqslant \sigma\, \mathrm{T}(\mathscr{Q}, X) \geqslant \sigma \max\left\{\varepsilon\delta, \frac{\Delta(\mathscr{Q}, X)}{\lambda}\right\} \geqslant 100\pi \sinh \rho.$$

Note that the second equation is deduced after using the geometric $C''(\lambda, \varepsilon)$-small cancellation condition. Therefore G, $\mathcal{X}$ and $\mathscr{Q}$ satisfy the hypothesis of the Small Cancellation Theorem (Lemma 2.27). We denote $K = \langle\langle H \mid (H, Y) \in \mathscr{Q}\rangle\rangle$ and $\bar{G} = G/K$. We denote by $\bar{A}$ the image of any set $A \subset G$ under the natural projection $\pi\colon G \twoheadrightarrow \bar{G}$.

The following lemma is the core of the proof of our main theorem. It brings together Theorem 4.8, Proposition 5.9 and Proposition 5.16.

Lemma 6.4. — *There exist $\beta \in (0,1)$ depending only on N with the following property. Let $U \subset G$ be a finite symmetric subset containing the identity such that $\mathrm{L}(U) \leqslant \pi \sinh 10^4\bar{\delta}$. Let $\Gamma = \langle U\rangle$. If Γ is non-elementary for the action on $\mathcal{X}$, then*

$$\omega(\bar{U}) \geqslant \beta\omega(U)$$

Proof. — We put

$$\beta = \sup_{\theta\in(0,1)} \inf\left\{\theta \cdot \frac{\log \frac{3}{2}}{\log(2c)}, 1-\theta\right\} \cdot \frac{1}{c}.$$

Let $U \subset G$ be a finite symmetric subset containing the identity such that $\mathrm{L}(U) \leqslant \pi \sinh 10^4\bar{\delta}$. Let $\Gamma = \langle U\rangle$ and assume that Γ is non-elementary for the action on $\mathcal{X}$. We are going to choose a power of U and apply Theorem 4.8 to that power for the (δ_0, N)-acylindrical action of G on the $\sigma\delta$-hyperbolic space $\mathcal{X}$. By assumption, we have

$$10^4 \cdot 200\delta_0 \leqslant \pi \sinh 10^4\bar{\delta}.$$

Since Γ is non-elementary, it follows from Lemma 6.1 that there exists $n \geqslant 1$ depending on U such that

$$10^4 \cdot 200\delta_0 < \mathrm{L}(U^n) \leqslant 2\pi \sinh 10^4\bar{\delta}. \tag{6.1}$$

Let $\Gamma' = \langle U^n\rangle$. Since U is symmetric and contains the identity, $U \subset U^n$. Therefore $\Gamma = \Gamma'$. The fact that Γ is non-elementary now implies that Γ' is non-elementary. Let $p \in \mathcal{X}$ be a point almost-minimizing the ℓ^∞-energy $\mathrm{L}(U^n)$. It follows from Theorem 4.8 that there exist a subset $S \subset G$ such that

(i) $S \subset U^{cn}$,

(ii) $|S| \geqslant \frac{1}{c}|U^n|$,

(iii) S is $200\delta_0$-reduced at p.

We are going to estimate $\omega(\bar{U})$. Let $r \geqslant 1$. Since U is symmetric and contains the identity, (i) implies

$$B_S(r) \subset U^{cnr}.$$

Let $F(\tau_2)$ be the set of τ_2-shortening-free words associated to U and $\mathcal{Q}$. We have

$$|\bar{U}^{cnr}| \geqslant |\bar{B}_S(r)| \geqslant |\bar{F}(\tau_0) \cap \bar{B}_S(r)|.$$

Further,

$$\mathrm{L}(S, p) \leqslant \mathrm{L}(U^{cn}, p) \leqslant c\,\mathrm{L}(U^n, p) \leqslant L_0,$$

where the first inequality is (i) and the second one is the triangle inequality. The third one is due to the upper bound of Equation 6.1, together with the fact that the point p is almost-minimizing the ℓ^∞-energy $\mathrm{L}(U)$. Hence we can apply Proposition 5.9 and Proposition 5.16 to obtain, respectively

$$|\bar{F}(\tau_2) \cap \bar{B}_S(r)| = |F(\tau_2) \cap B_S(r)|, \quad \text{and} \quad |F(\tau_2) \cap B_S(r)| \geqslant \left[\frac{1}{2}(2|S| - 1)\right]^r.$$

Applying Fekete's Subadditive Lemma,

$$|U^n| \geqslant e^{n\omega(U)}.$$

Together with (ii), this implies

$$2|S| - 1 \geqslant |S| \geqslant \frac{1}{c} e^{n\omega(U)}.$$

Combining our estimations, we deduce

$$|\bar{U}^{cnr}| \geqslant \max\left\{ \left[\frac{1}{2}(2|S| - 1)\right]^r, \left[\frac{1}{2c} e^{n\omega(U)}\right]^r \right\}. \tag{6.2}$$

We have,

$$\omega(\bar{U}) = \limsup_{r \to \infty} \frac{1}{cnr} \log |\bar{U}^{cnr}|.$$

Let $\theta \in (0,1)$. Consider the positive number

$$\gamma = \frac{\log 2c}{\theta \omega(U)}.$$

► If $n \leqslant \gamma$, we use the first bound of Equation 6.2 to obtain

$$\omega(\bar{U}) \geqslant \frac{1}{cn} \cdot \log \left[\frac{1}{2}(2|S| - 1)\right].$$

Since $n \leqslant \gamma$, we have $\frac{1}{n} \geqslant \frac{1}{\gamma}$. Further, $|S| \geqslant 2$. Consequently,

$$\omega(\bar{U}) \geqslant \theta \cdot \frac{\log \frac{3}{2}}{\log 2c} \cdot \frac{1}{c} \cdot \omega(U).$$

► If $n \geqslant \gamma$, we use the second bound of Equation 6.2 to obtain

$$\omega(\bar{U}) \geqslant \frac{1}{c}\left(\omega(U) - \frac{1}{n}\log 2c\right).$$

Since $n \geqslant \gamma$, we have $\frac{1}{n} \leqslant \frac{1}{\gamma}$. Consequently,

$$\frac{1}{n}\log 2c \leqslant \theta\omega(U).$$

Therefore,

$$\omega(\bar{U}) \geqslant (1-\theta) \cdot \frac{1}{c} \cdot \omega(U).$$

Finally, combining the cases $n \leqslant \gamma$ and $n \geqslant \gamma$, we obtain:

$$\omega(\bar{U}) \geqslant \beta\omega(U).$$

□

Theorem 6.5 (Theorem 1.2 (i))**. —** *Let $\xi > 0$. If G has ξ-uniform uniform exponential growth, then every geometric $C''(\lambda, \varepsilon)$-small cancellation quotient of G has ξ'-uniform uniform exponential growth. The constant ξ' depends only on ξ and N.*

Proof. — Let $\xi > 0$. Assume that G has ξ-uniform uniform exponential growth. Let $\bar{U} \subset \bar{G}$ be a finite symmetric subset containing the identity and denote $\bar{\Gamma} = \langle \bar{U} \rangle$. Recall that $\mathscr{V}$ stands by the set of apices of the cone-off space $\dot{\mathcal{X}}_\rho(\mathscr{Q}, X)$. There are two cases:

Case 1. *There exist $\bar{v} \in \bar{\mathscr{V}}$ such that $\bar{U}$ is contained in* $\mathrm{Stab}(\bar{v})$.

Let $v \in \mathscr{V}$ be a preimage of $\bar{v}$. Let $(H, Y) \in \mathscr{Q}$ such that v is the apex of the cone $Z(Y)$. The natural projection $\pi\colon G \twoheadrightarrow \bar{G}$ induces an isomorphism $\mathrm{Stab}(Y)/H \xrightarrow{\sim} \mathrm{Stab}(\bar{v})$ (Lemma 2.27 (iii)). Since the moving family $\mathscr{Q}$ is loxodromic, H has finite index in $\mathrm{Stab}(Y)$. Hence $\bar{\Gamma}$ is finite, in particular virtually nilpotent.

Case 2. *The set $\bar{U}$ is not contained in* $\mathrm{Stab}(\bar{v})$, *for every $\bar{v} \in \bar{\mathscr{V}}$.*

The quotient space $\bar{\mathcal{X}}_\rho$ is $\bar{\delta}$-hyperbolic (Lemma 2.27 (i)) and the action of $\bar{\Gamma}$ on $\bar{\mathcal{X}}_\rho$ is acylindrical (Lemma 2.35). Then $\bar{\Gamma}$ falls exactly in one of the following three situations (Lemma 2.22):

(a) *$\bar{\Gamma}$ is elliptic, or equivalently one (hence any) orbit of $\bar{\Gamma}$ is bounded.* Since the set $\bar{U}$ is not contained in $\mathrm{Stab}(\bar{v})$, for every $\bar{v} \in \bar{\mathscr{V}}$, there exists an elliptic subgroup $E \subset G$ for the action of G on $\mathcal{X}$ such that the natural projection $\pi\colon G \twoheadrightarrow \bar{G}$ induces an isomorphism $E \xrightarrow{\sim} \bar{\Gamma}$ (Lemma 2.31). Since G has ξ-uniform uniform exponential growth, the subgroup E is either virtually nilpotent or has ξ-uniform exponential growth. In combination with the isomorphism $F \xrightarrow{\sim} \bar{\Gamma}$, we deduce that $\bar{\Gamma}$ either is virtually nilpotent or has ξ-uniform exponential growth.

(b) $\bar{\Gamma}$ *is loxodromic, or equivalently* $\bar{\Gamma}$ *is virtually cyclic and contains a loxodromic element.* Then $\bar{\Gamma}$ is virtually nilpotent.

(c) $\bar{\Gamma}$ *is non-elementary, or equivalently* $\bar{\Gamma}$ *contains a free group* $\mathbf{F}_2$ *of rank* 2 *and one (hence any) orbit of* $\mathbf{F}_2$ *is unbounded.* There are two subcases:

(E1) *Large energy*: $\mathrm{L}(\bar{U}) > 10^4\bar{\delta}$.

Then $\omega(\bar{U}) \geqslant \frac{1}{10^3}\log 2$ (Lemma 2.22 and Lemma 2.23). Note that here we do not require any control over the parameters of the acylindrical action of $\bar{\Gamma}$ on $\bar{\mathcal{X}}_\rho$.

(E2) *Small energy*: $\mathrm{L}(\bar{U}) \leqslant 10^4\bar{\delta}$.

Since $\bar{U}$ is not contained in $\mathrm{Stab}(\bar{v})$, for every $\bar{v} \in \bar{\mathscr{V}}$, and $10^4\bar{\delta} \leqslant \rho/5$, there exists a pre-image $U \subset G$ of $\bar{U}$ of energy $\mathrm{L}(U) \leqslant \pi \sinh 10^4\bar{\delta}$ (Lemma 2.32). Without loss of generality, we may assume that U is symmetric and contains the identity. Since $\bar{\Gamma}$ is non-elementary for the action on $\bar{\mathcal{X}}_\rho$, the subgroup Γ is non-elementary for the action on $\mathcal{X}$ (Lemma 2.29). According to Lemma 6.4, there exists $\beta \in (0,1)$ depending on N such that $\omega(\bar{U}) \geqslant \beta\omega(U)$. Since G has ξ-uniform uniform exponential growth and Γ is non-elementary, we have $\omega(U) \geqslant \xi$. Therefore, $\omega(\bar{U}) \geqslant \beta\xi$. This completes the proof of our theorem.

□

Theorem 6.6 (Theorem 1.2 (ii))**. —** *Let $\xi > 0$. If there exists a geometric $C''(\lambda, \varepsilon)$-small cancellation quotient of G that has ξ-uniform uniform exponential growth, then G has ξ'-uniform uniform exponential growth. The constant ξ' depends only on ξ.*

Proof. — Let $\xi > 0$. Assume that $\bar{G}$ has ξ-uniform uniform exponential growth. Let $U \subset G$ be a finite symmetric subset containing the identity and denote $\Gamma = \langle U \rangle$. Then Γ falls exactly in one of the following three situations (Lemma 2.22):

(a) Γ *is elliptic, or equivalently one (hence any) orbit of* Γ *is bounded.* The projection $\pi\colon G \twoheadrightarrow \bar{G}$ induces an isomorphism $\Gamma \xrightarrow{\sim} \bar{\Gamma}$ (Lemma 2.30). Since $\bar{G}$ has ξ-uniform uniform exponential growth, the subgroup $\bar{\Gamma}$ is either virtually nilpotent or has ξ-uniform exponential growth. In combination with the isomorphism $\Gamma \xrightarrow{\sim} \bar{\Gamma}$, we deduce that Γ is either virtually nilpotent or has ξ-uniform exponential growth.

(b) Γ *is loxodromic, or equivalently* Γ *is virtually cyclic and contains a loxodromic element.* Then Γ is virtually nilpotent.

(c) Γ *is non-elementary, or equivalently* Γ *contains a free group* $\mathbf{F}_2$ *of rank* 2 *and one (hence any) orbit of* $\mathbf{F}_2$ *is unbounded.* There are two subcases:

(E1) *Large energy*: $\mathrm{L}(U) > 10^4\delta_0$.

Then $\omega(U) \geqslant \frac{1}{10^3}\log 2$ (Lemma 2.22 and Lemma 2.23). Note that here we do not require any control over the parameters of the acylindrical action of Γ on $\mathcal{X}$.

(E2) *Small energy*: $\mathrm{L}(U) \leqslant 10^4\delta_0$.

By definition, $\omega(U) \geqslant \omega(\bar{U})$. Since Γ is non-elementary for its action on $\mathcal{X}$, we have $\omega(U) > 0$. Since $10^4\delta_0 \leqslant \pi \sinh 10^4\bar{\delta}$, it follows from Lemma 6.4 that $\omega(\bar{U}) > 0$. In particular $\bar{\Gamma}$ is not virtually nilpotent. Since $\bar{G}$ has ξ-uniform uniform exponential growth, we deduce that $\omega(\bar{U}) \geqslant \xi$. Therefore, $\omega(U) \geqslant \xi$.

□

7. References

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