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ERGODICITY OF THE GEODESIC FLOW IN NEGATIVE CURVATURE

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«*Cómpre romper a marcha pola mesma estrada que fagamos cos nosos pasos e afrontar nela unha peregrinaxe sen chegada, porque en cada relanzo do camiño agárdanos unha voz que nos berra: ¡Máis alá!*»

Manuel Antonio e Álvaro Cebreiro
Manifesto "¡Máis Alá!", 1922

Declaration

I certify that this report has been written by me and it is a record based on my personal study of other sources.

Ergodicity of the Geodesic Flow in Negative Curvature

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Abstract

Consider a proper, $CAT(-1)$ space X compactified by its Gromov boundary at infinity ∂X. Let Γ be a non-elementary group acting by isometries and properly discontinuously on X. We construct an invariant measure m_{BM} for the geodesic flow on the orbispace of geodesics SX_Γ. This measure is built from a family of measures on ∂X which are based on the distribution of the points in a Γ-orbit. Then, we apply this construction to connect the action of Γ on ∂X and the geodesic flow on SX_Γ.

Introduction

Geodesic flows have been cornerstone in the development of the theory of dynamical systems: most of the results were obtained first for the case of the geodesic flow before being generalised to a wider range of dynamics. The study of the geodesic flow for a (0) closed manifold of negative curvature ([Bal95], appendix) is by now classical and overexploited. Modern directions in its study include (1) weakening the compactness assumption ([Cou09]), (2) weakening the hypothesis on the curvature ([DOP00], [LP16]) or (3) weakening the assumption that the underlying space is a Riemannian manifold, i.e., considering nice orbispaces made of $CAT(-1)$ and *Hadamard spaces* ([Rob03], [Lin18]). In addition, one may notice that those manifolds verify the following fact: they are orbispaces made from some «model space» with non-positive curvature and from some group acting by isometries and properly discontinuously on it.

The main purpose on the creation of these notes was to learn something useful about ergodic theory in negative curvature. In order to be successful in this project, the underlining aim was reduced to the following one : after finishing this notes and having some rest, one could be able to take [Sul79] and read it harmoniously if one excludes section 7. As a result of this, we have organised the text as follows: on the first chapter we set up our model space and its orbispaces, which are a generalisation of the n-hyperbolic spaces and hyperbolic manifolds. The hyperbolic case uses from time to time a simply transitive hypothesis that might make unclear for the non-expert the potential aspects of metric geometry since this property is more related to symmetric spaces. I decided not to tackle both cases at the same time. The second chapter is about the construction of the geodesic flow and the Patterson-Sullivan measure at infinity : this summons ergodic theory and allows one to talk about divergence and convergence type groups, as well as to establish a dichotomy between ergodicity at the boundary and at the geodesic space. Finally the last chapter agrees with a big part of [Sul79] despite being based on [Rob03]. The appendix is also important but Hausdorff dimension is not where I have decided to put most of my efforts due to time constraints. I would suggest to go to the appendix only once having gone through chapters 1, 2,3 and not before. Last but not least, remark that there is no original proof of result on these notes.

To visually sum up the message, all this means that after reading most of the notes you would be able to understand the following statement:

Theorem 0.1
Let μ be a Γ-invariant, α-conformal density. Then, we have two complementary situations (A) and (B). In each situation, the statements from (i) to (iv) are equivalent:

(A) (i) $\sum_{\gamma\in\Gamma} e^{-\alpha\, d(o,\gamma o)} = \infty$

(ii) $\mu_x(\Lambda_c(\Gamma)) = 1$

(iii) $(SX_\Gamma, m_{BM}, g_\Gamma^t)$ is conservative and ergodic.

(iv) $(\partial X^2, M_{BM}, \Gamma)$ is conservative and ergodic.

(B) (i) $\sum_{\gamma\in\Gamma} e^{-\alpha\, d(o,\gamma o)} < \infty$

(ii) $\mu_x(\Lambda_c(\Gamma)) = 0$

(iii) $(SX_\Gamma, m_{BM}, g_\Gamma^t)$ is dissipative and non-ergodic.

(iv) $(\partial X^2, M_{BM}, \Gamma)$ is dissipative and non-ergodic.

Acknowledgements. I would like to take the time to express my great appreciation to *Rémi Coulon*. His great availability, openness and uncountable advice introduced me to a field which is full of variety and beautiful mathematical objects.

Contents

Notation

Here there is only some notation :

$\overline{B}(x,r)$	closed ball of centre x and radius r.
d_x	metric on ∂X, see section 1.2.3.
$\triangle$	the diagonal of a set *OR* a triangle.
$\mathbb{H}^n$	the n-dimensional hyperbolic space.
$Homeo(X)$	the group of homeomorphisms of X.
$Int(F)$	interior set of F.
$Iso(X)$	the group of isometries of X.
$\mathcal{L}^n$	the Lebesgue measure on $\mathbb{R}^n$.
o	a base point of a proper, $CAT(-1)$ space.
$\emptyset$	the diameter of a set.
Ω	a topological space or a metric space.
$\mathcal{O}_r(x,y)$	a shadow at infinity.
$\mathcal{P}(\Omega)$	the set of parts of Ω.
∂X	boundary at infinity of X.
$\partial^2 X$	$\partial X \times \partial X - \triangle$.
$\mathbb{R}$	the set of real numbers.
X	a metric space or a proper, $CAT(-1)$ space.
$\|x-y\|$	the absolute value on $\mathbb{R}$.
$r \gg 0$	for r big.

Chapter 1

$CAT(-1)$ Boundaries and Orbispaces

We state here some basic elements of metric geometry, issued from the notes of [Bal95], [BH99] and [Bou95]. We also recommend have a look to the first chapter of [Dal07] so that you can gain intuition on the hyperbolic plane case $\mathbb{H}$ if you feel so.

1.1 Definitions

1.1.1 Geodesics

Set (X,d) a metric space.

A `path` on X is a continuous map $v\colon I \to X$ where I is some non-degenerated interval of $\mathbb{R}$. If $I = [a,b]$ or $I = [0,\infty)$ or $I = \mathbb{R}$ we say that v is a `segment` or a `ray` or a `line`, respectively.

Definition 1.1
The `length` $L(v)$ of a segment $v\colon [a,b] \to X$ is defined as

$$L(v) := sup \sum_{i=0}^{k-1} d(v(t_i), v(t_{i+1})))$$

where the supremum is taken over all subdivisions

$$a = t_0 < t_1 < \cdots < t_k = b$$

Remark 1.2
If $\varphi\colon [a',b'] \to [a,b]$ is an homeomorphism, then $L(v \circ \varphi) = L(v)$, .i.e., re-parametrised segments have the same length; and there is a unique parametrisation proportional to arc-length.

Directly from the definition, it follows that if a segment $v\colon [a,b] \to X$ connects two points $x, y \in X$ (i.e., if $v(a) = x$ and $v(b) = y$) then,

$$L(v) \geq d(x,y)$$

A `geodesic` is a path $v\colon I \to X$ that is also an isometry; in other words, v is a `geodesic` if

$$\forall\, t, t' \in \mathbb{R} \quad d(v(t), v(t')) = |t - t'|$$

If we specialise $I = [a,b]$ or $I = [0,\infty)$ or $I = \mathbb{R}$ we say that v is a `geodesic segment` or a `geodesic ray` or a `geodesic line`, respectively.

Remark 1.3
According to the context, we may also refer to the image of v as a geodesic.

Thus, a geodesic segment $v\colon [a,b] \to X$ connecting $x, y \in X$ is a «minimising» path connecting them since

$$L(v) = \sup \sum_{i=0}^{k-1} |t_i - t_{i+1}| = \sup |a-b| = d(x,y)$$

Nevertheless, without any further assumptions in our metric space X we cannot guarantee the existence, not to say the uniqueness, of a geodesic connecting two arbitrary points of it.

Definition 1.4 (I) X is said to be a `geodesic space` if $\forall\, x, y \in X$ there exists a geodesic segment connecting x to y.
(II) X is said to be a `uniquely geodesic space` if $\forall x, y \in X$ there exists a unique geodesic segment connecting x to y.

Notation 1.5
Despite lack of uniqueness in case of existence, we will denote by $[xy]$ a geodesic segment connecting $x, y \in X$, assuming that we have chosen a specific one.

A `geodesic triangle` $\triangle$ is a set $[xy] \cup [xz] \cup [yz]$, where $[xy]$, $[xz]$, $[yz]$ are three chosen geodesic segments connecting pairwise three points $x, y, z \in X$.

1.1.2 Our «Model Space»

Set (X, d_X) a geodesic space.

We introduce now two families of metric spaces that generalise the hyperbolic plane $\mathbb{H}$: the δ–hyperbolic spaces and the $CAT(-1)$ spaces.

Gromov's product : Given $x, y, z \in X$, the quantity

$$\langle x, y\rangle_z := \frac{1}{2}[d(x,z) + d(y,z) - d(x,y)]$$

is called `Gromov's product` of x, y, z.

Definition 1.6 (δ-Hyperbolic space)
Let $\delta \leq 0$. X is δ`-hyperbolic` if every $x, y, z \in X$ satisfy `Gromov's inequality` :

$$\langle x, y\rangle_z \geq \min\{d(x,z), d(y,z)\} - \delta$$

In that case, every geodesic triangle $[xy] \cup [xz] \cup [yz]$ is 4δ`-thin`, i.e.,

$$\forall\, p \in [xy] \quad d_X\,(p, [xz] \cup [yz]) \leq 4\delta$$

and similarly for points on the other edges.

Remark 1.7

(1) A definition with Gromov's inequality is necessary since we will use a extended version along the proof of the Shadow lemma (theorem 2.23).

(2) $\langle x, y\rangle_z \leq d_x(z, [xy]) - \delta$.

Given a geodesic triangle $\triangle = [xy] \cup [xz] \cup [yz]$ on X, we can associate it to another triangle $\overline{\triangle} = [\overline{x}\,\overline{y}] \cup [\overline{x}\,\overline{z}] \cup [\overline{y}\,\overline{z}]$ (perimeter condition concerns) on the hyperbolic plane $(\mathbb{H}, d_{\mathbb{H}})$ called the `comparison triangle` of $\triangle$ and $\overline{\triangle}$ is unique up to isometries.

Definition 1.8
A geodesic triangle $\triangle$ of X satisfies the $CAT(-1)$`-inequality` if for every $a, b \in \triangle$ their comparison points $\overline{a}, \overline{b} \in \overline{\triangle}$ satisfy the inequality

$$d_X(a,b) \leq d_{\mathbb{H}}(\overline{a}, \overline{b})$$

X is a $CAT(-1)$`-space` if every geodesic triangle satisfy the $CAT(-1)$ inequality.

Proposition 1.9 (Properties ;[BH99], proposition 1.4)
A $CAT(-1)$ space is $\cdots$

(i) Uniquely geodesic.

(ii) Contractible and therefore simply connected (similarly to model manifolds).

(iii) δ-hyperbolic.

Proof. This follows easily by comparison with $\mathbb{H}$. ■

Example 1.10 (Classical)
Among the species, we find $CAT(-1)$ spaces such as:

(1) Real trees.

(2) The hyperbolic spaces $\mathbb{H}^n$.

(3) Simply connected Riemannian manifolds with negative curvature [i.e., with sectional curvatures ≤ -1].

More generally, Riemannian manifolds with negative curvature ≤ -1 are locally $CAT(-1)$.

Last but not least, we need the following adjective:

Definition 1.11
A metric space (X, d) is said to be `proper` if every closed ball is compact.

As an immediate consequence, a proper, metric space X is locally compact and complete. In fact, the inverse is also true on the context of geodesic spaces, due to the *Hopf-Rinow* ([GHL04], section $2.C.5$ or the lovely [dC92], chapter 7), a fundamental result connecting Riemannian and metric geometry which tells us that proper geodesic spaces are «the chosen ones». Notice however that in the non-Riemannian setting completeness of X does not imply that every geodesic segment can be extended to a geodesic line, so X may not be geodesically complete.

Observe that X is also σ-compact (or, equivalently, separable in the context of locally compact metric spaces) : if you take the reunion of closed balls centred in some point and with natural radius you will cover X countably by compact sets. We will use this point on section 3.1.

Example 1.12
A real tree with a «finite» branch is a proper geodesic space and it is not geodesically complete.

On these notes, we are interested on proper, $CAT(-1)$ spaces. This kind of space with be our «model space».

1.2 Technology

In this section we introduce the boundary at infinity, orbispaces and some tools to work on it.

1.2.1 Gromov's Boundary

Here we give very quickly the statements related to a particular boundary at infinity of a proper, $CAT(-1)$ space X. This boundary is, for instance, homeomorphic to the sphere on the case $X = \mathbb{H}^n$.

Definition 1.13
We say that two geodesic rays v_1, v_2 of X are `asymptotic` if

$$\exists K > 0 \ | \ \forall t \geq 0 \ \ d(v_1(t), v_2(t)) \leq K$$

and we write $v_1 \sim v_2$.

For $x \in X$, denote $\mathcal{R}_x = \{v \colon X \to \mathbb{R} \ \ \textit{geodesic ray} \mid v(0) = x\}$. It is straightforward that $\sim$ defines an equivalence relation on $\mathcal{R}_x$. Also, observe the following:

Remark 1.14
$\forall x, x' \in X$ we have a bijection :

$$\mathcal{R}_x/\sim \leftrightarrow \mathcal{R}_{x'}/\sim$$

From now on $o \in X$ will be a base point so that we can write $\mathcal{R} = \mathcal{R}_o$.

Notation 1.15
If v is a geodesic line, $v(-\infty)$ will denote the class of $v_{|[0,\infty)}(-t)$ and $v(+\infty)$ will denote the class of $v_{|[0,\infty)}(t)$. If v is just a geodesic ray, $v(+\infty)$ will also denote its class.

Definition 1.16
The `Gromov boundary` or boundary at infinity of X is the quotient set:

$$\partial X := \mathcal{R}/\sim$$

The following lemma is handy when proving existence result related to geodesics.

Lemma 1.17 (Arzelà-Ascoli; [BH99] proposition I.3.10)
Set Ω_1 a separable metric space and Ω_2 a compact metric space. Then, every sequence of equicontinuous maps f_n has a subsequence that converges (uniformly on compact sets) to a continuous map $f \colon \Omega_1 \to \Omega_2$.

Proposition 1.18 ([BH99], lemmas III.H.3.1 and III.H.3.2)
Set X a proper, $CAT(-1)$ space.

(i) For every $x \in X$ and $\xi \in \partial X$ there exists a geodesic ray v, unique up to re-parametrisation, such that $v(0) = x$ and $v(+\infty) = \xi$.

(ii) (*Visibility property*). For every $\xi, \eta \in \partial X$ there exists a geodesic line v, unique up to re-parametrisation, such that $v(-\infty) = \xi$ and $v(+\infty) = \eta$.

Notation 1.19
If $\xi, \eta \in \partial X$ and $x \in X$ we write $[x\xi)$ for the geodesic ray from x to ξ and $(\xi\eta)$ for a geodesic line from ξ to η.

Compactification. If $\overline{X} = X \cup \partial X$, what to say about the topology that makes $\overline{X}$ a compact space? Well, there exists a topology ([BH99], definition III.H.3.5) such that $\overline{X}$ is compact (application: theorem 2.15) and ∂X is compact (application: the appendix). In fact, this compactification is homeomorphic to a closed disk in case $X = \mathbb{H}$.

1.2.2 Actions by Isometries and Orbispaces

Set (Ω, d) a proper, metric space.

An `action by isometries` of a group Γ on Ω is a group homomorphism

$$\rho\colon \Gamma \to Iso(\Omega)$$

This implies that

$$\forall \gamma \in \Gamma \ \ \forall x, y \in \Omega \quad d(\rho(\gamma)x, \rho(\gamma)y) = d(x,y)$$

Whenever we talk about an action we assume that it is by isometries. The unique exception will be the additive group $\mathbb{R}$ in definition 2.2.

Notation 1.20
If $\gamma \in \Gamma$ and $x \in \Omega$ we write $\rho(\gamma)x = \gamma x$ by abuse of notation. The action could be written somewhere as $\Gamma \curvearrowright \Omega$.

Among the actions by isometries of our interest one could find the following; notice however that they are usually defined for more general spaces, though:

Definition 1.21
The action $\Gamma \curvearrowright \Omega$ is $\cdots$

(I) `Free` if

$$\forall \gamma \in \Gamma - \{Id\} \ \ \forall x \in \Omega \quad \gamma x \neq x$$

(II) `Properly discontinuous` for every compact $K \subset \Omega$ we have

$$\#\{\gamma \in \Gamma \mid \gamma K \cap K \neq \emptyset\} < \infty$$

(III) `co-compact` if there exists a compact subset K_0 such that

$$\Omega \subset \bigcup_{\gamma \in \Gamma} \gamma K_0$$

(IV) `Ergodic` if Ω is also a measure space $(\Omega, \mathcal{B}, M)$ and

$$A \in \mathcal{B} \mid \forall \gamma \in \Gamma,\ \gamma A = A \ \Rightarrow\ [M(A) = 0\ \vee\ M(\Omega - A) = 0]$$

Remark 1.22

(a) In (II) and (III) we can substitute K by $\overline{B}(x, r)$ $\forall x \in \Omega, \forall r > 0$ and K_0 for some $\overline{B}(x_0, r_0)$, respectively.

(b) Moreover, if the action of Γ on Ω is properly discontinuous, we obtain

$$\forall x, y \in \Omega \ \ \forall r > 0 \quad \#\{\gamma \in \Gamma \mid \gamma y \in B(x, r)\} < \infty$$

and since $\Omega = \bigcup_{k=0}^{\infty} [B(o, k+1) - B(o, k)]$ we see that Γ is a countable set.

This following result permits to metrify the orbit space

$$\Omega_\Gamma = \Gamma \backslash \Omega := \{\Gamma x \mid x \in \Omega\}$$

exactly as it is done in the case of hyperbolic manifolds, for instance. Observe that Ω_Γ is the quotient space defined by the equivalence relation on Ω

$$x \sim y :\Leftrightarrow \exists \gamma \in \Gamma \quad | \quad \gamma x = y$$

Theorem 1.23 ([Hai10], proposition 3.11)
Let Γ be a group acting properly discontinuously on Ω. Define the map

$$d_{\Omega_\Gamma} \colon \Omega_\Gamma \times \Omega_\Gamma \to \mathbb{R}_{\geq 0} \quad | \quad d_{\Omega_\Gamma}(\Gamma x, \Gamma y) := \inf_{\gamma \in \Gamma} d(x, \Gamma y)$$

Then,

(i) d_{Ω_Γ} is a metric on Ω_Γ suitable for the quotient topology.

(ii) The projection $\pi \colon \Omega \to \Omega_\Gamma$ is a local isometry if and only if the action of Γ is free.

Example 1.24
The set of orbispaces made of proper, $CAT(-1)$ spaces and a group acting properly discontinuously on it include all complete manifolds of negative curvature (it is the *Cartan-Hadamard* theorem [Bal95]).

We close this subsection with the following device used on proposition 3.14 :

Definition 1.25 (Fundamental Domain)
Let $\Gamma \curvearrowright \Omega$. A closed subset $F \subset \Omega$ is called a `fundamental domain` of Γ in Ω if:
(I) $\Omega = \bigcup_{\gamma\in\Gamma} \gamma F$
(II) $Int(F) \cap Int(\gamma F) = \emptyset \quad \forall \gamma \in \Gamma - \{Id\}$

Remark 1.26
$\Gamma \backslash F$ is homeomorphic to Ω_Γ.

1.2.3 Busemann Functions, Metrification of ∂X and Horospheres

Set X a proper, $CAT(-1)$ space.

Busemann Functions.

Proposition-Definition 1.27
Let v be a geodesic ray of X. A `Busemann function` is a map

$$b_v : X \to \mathbb{R} \ , \ x \mapsto b_v(x) := \lim_{t\to\infty}[d(x, v(t)) - t]$$

This a well-defined function and independent of the class of v in ∂X.

Proof. Well-definition. First observe that, by the triangle inequality, for all $t, t' \in \mathbb{R}$,

$$d(v(t), v(t')) - d(x, v(t)) \le d(x, v(t)) \le d(x, v(t')) + d(v(t), v(t'))$$

This implies :

1. The function $t \in [0,\infty) \mapsto d(x, v(t)) - t \in \mathbb{R}$ is non-increasing:

$$d(x, v(t)) - t \le d(x, v(t')) - t'$$

2. The function $t \in [0,\infty) \mapsto d(x, v(t)) - t \in \mathbb{R}$ is bounded below by a constant:

$$t - t' \le d(x, v(t)) + d(x, v(t)) \Rightarrow d(x, v(t)) - t \ge -d(x, v(0))$$

Therefore, b_v is well-defined.
Independence. Finally, if $v, v \in \xi \in \partial X$ then recall that $v = v'$ since our definition of ∂X assumes that $v(0) = v'(0) = o$ and we have uniqueness of this geodesic ray due to proposition 1.18.

■

Remark 1.28
In the prove of the independence it is essential to use the $CAT(-1)$ inequality and not the δ-Hyperbolicity, otherwise we would only obtain $d(v'(t), v) < K$ for some constant $K > 0$.

Definition 1.29
Let $x, y \in X$ and $\xi \in \partial X$. The `Busemann cocycle` of x, y based on ξ is

$$\beta_\xi(x, y) = b_v(x) - b_v(y)$$

where v is a geodesic ray belonging to the class of ξ.

We state some immediate properties that could be useful beyond this page:

Proposition 1.30 (Properties)
For every $\xi \in \partial X$ and $x, y, z \in X$:

(i) $\beta_\xi(x, y) = -\beta_\xi(y, x)$

(ii) $\beta_\xi(x, z) = \beta_\xi(x, y) + B_\xi(y, z)$

(iii) $|\beta_\xi(x, y)| \leq d(x, y)$.

(iv) $\beta_\xi(x, y) = d(x, y)$ $[-d(x, y)]$ if and only if $y \in [x\xi)$ $[x \in [y\xi)]$.

(v) If Γ acts by isometries on X and $\gamma \in \Gamma$ then, $\beta_{\gamma\xi}(\gamma x, \gamma y) = \beta_\xi(x, y)$

An essential property used on 2.15 is the following:

Proposition 1.31 ([Hai10], proposition 2.6)
$\partial X \times X \times X \longrightarrow \mathbb{R}$, $(\xi, x, y) \mapsto B_\xi(x, y)$ is continuous.

Remark 1.32 (Extended Gromov's product)
Another important fact of Busemann functions is that one can extend Gromov's product for $z \in X$ and $\xi, \eta \in \partial X$ by defining

$$\langle \xi, \eta \rangle_z = \lim_{n\to\infty} \langle x_n, y_n \rangle_z$$

with $(x_n)_n \in \xi$ and $(y_n)_n \in \eta$.

This extended product not only verifies an extended Gromov's inequality but also, if $o \in (\eta\xi)$,

$$\langle \xi, \eta \rangle_z = \frac{1}{2}(\beta_\xi(z, o) + \beta_\eta(z, o))$$

Metrification of ∂X. There exists a family of metrics on ∂X verifying a property called «conformality»; nevertheless, the definition of this word would be unimportant for the rest of our exposition even you will see appear it again on chapter 3. Conformality is a mean, not an end (on these notes).

Theorem 1.33 ([Bou95], 2.5.1)
Let $x \in X$ and define for all $\xi, \eta \in \partial X$:

$$d_x(\xi,\eta) = e^{-\langle \xi,\eta\rangle_x} \qquad \text{if } \xi \neq \eta$$
$$d_x(\xi,\eta) = 0 \qquad \text{otherwise}$$

Then, d_x is a metric on ∂X.

Example 1.34
One application of conformality is on the original proof's of *Mostow's rigidity theorem* [Mos68]: two closed hyperbolic manifolds have the same fundamental group if and only if they are isometric. If Γ is the fundamental group of one of such manifolds, say $\mathcal{M}$, a particular invariant is $dim(\Lambda(\Gamma))$, the Hausdorff dimension (definition A.4, appendix) of the limit set $\Lambda(\Gamma)$ (definition 1.36 below). And one could find that the Hausdorff dimension sometimes coincides or is related to other invariants such as:

- δ_Γ, the critical exponent of the Poincaré series of Γ (see section 3.1 for a definition).
- $\lambda_0(M)$, the bottom of the spectrum of the *Laplacian* on $\mathcal{M}$ ([Sul87]).

If you feel curious, check the *appendix* for one connection between $dim(\Lambda(\Gamma))$ and δ_Γ on the co-compact case.

Horospheres. To close this subsection we introduce horospheres, expecting to use them later somewhere in section 2.1 in order to define the stable and unstable foliations as in the case $X = \mathbb{H}$:

Definition 1.35
An `horosphere` $H_{y,\xi}$ centred at $\xi \in \partial X$ and passing through $y \in X$ is the set

$$\{z \in X \mid \beta_\xi(y,z) = 0\}$$

On section 2.1 we will give a more dynamic characterisation of horospheres.

1.2.4 The Radial Limit Set

Set X a proper, $CAT(-1)$ space and Γ a group acting properly discontinuously on X.

All the orbits Γx have the same accumulation set on ∂X. This is an immediate application of the property that if $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are two sequences with bounded distance from each other and $x_n \xrightarrow{n\to\infty} \xi \in \partial X$ then, $y_n \xrightarrow{n\to\infty} \xi$ as well by definition of this topology.

Definition 1.36
The `limit set` $\Lambda(\Gamma)$ of Γ is the accumulation set on ∂X of some Γ−orbit on X :

$$\Lambda(\Gamma) := \overline{\Gamma o} \cap \partial X$$

If $\#\Lambda(\Gamma) < \infty$ we say that Γ is an `elementary group`; otherwise if $\#\Lambda(\Gamma) = \infty$ we say Γ is a `non-elementary group`.

Remark 1.37

(1) If Γ is non-elementary, there are infinitely many couples of distinct points $(\xi_-, \xi_+) \in \partial X \times \partial X - \triangle$ which are fixed by some $\gamma \in \Gamma$ (those $\gamma \in \Gamma$ are hyperbolic isometries).

(2) $\Lambda(\Gamma)$ is the smallest non-empty, Γ-invariant, closed subset of ∂X.

Definition 1.38
A point $\xi \in \Lambda(\Gamma)$ is `radial` if for some $o \in X$ there exists $C > 0$ and $(\gamma_n)_{n\geq 0} \subset \Gamma$ such that the sequence $(\gamma_n o)_{n\geq 0}$ converges to ξ and $d(\gamma_n o, [o\xi)) \leq C$. The `radial limit set` $\Lambda_c(\Gamma)$ is the set of radial points.

Remark 1.39
A fixed point at infinity of a hyperbolic isometry is radial. Thus, if Γ is non-elementary $\Lambda_c(\Gamma)$ is infinite.

Chapter 2

Measures at Infinity

We define the geodesic flow for a proper, $CAT(-1)$ space and endow it with an invariant measure.

This chapter relies on [Bou95] for section 2.1; [Sul79] and [Nic89] for section 2.2; some comments of [Kai91] for section 2.3 and [Hai10] for section 2.4.

2.1 Geodesic Flow

Set X a proper, CAT(-1) space and Γ a group acting by isometries with a properly discontinuous action.

2.1.1 Definitions

Define SX the set of geodesic lines of X:

$$SX := \{v : \mathbb{R} \to X \textit{ isometry}\}$$

and endow it with the following metric

$$d_{SX}(v_1, v_2) = \int_{\mathbb{R}} d_X(v_1(t), v_2(t)) \frac{e^{-|t|}}{2}\, dt$$

It is clear that Γ acts by isometries on SX : this is induced by d_X. In regards to the proper discontinuity of $\Gamma \curvearrowright SX$, it is almost immediate. Thus, the orbit space

$$SX_\Gamma := \Gamma \backslash SX$$

is also a metrisable space after proposition 1.23.

Example 2.1
If $X = \mathbb{H}^n$ and $\Gamma < Iso(\mathbb{H}^n)$ acts freely and properly discontinuously, then SX is homeomorphic to the unit tangent bundle $T^1\mathbb{H}^n$ of $\mathbb{H}^n$ and SX_Γ is homeomorphic to the unit tangent bundle $\Gamma\backslash T^1\mathbb{H}^n$ of the hyperbolic manifold $\Gamma\backslash\mathbb{H}^n$ where $\Gamma \curvearrowright T^1\mathbb{H}$ is the induced action.

Definition 2.2
Let Ω be a topological space. A `flow` on Ω is an $\mathbb{R}$-action on Ω, i.e., a group homomorphism

$$\varphi\colon \mathbb{R} \to Homeo(X)$$

Notation 2.3
We write $\varphi(t) = \varphi^t \ \forall\, t \in \mathbb{R}$.

The `geodesic flow of` SX is the flow g^t on SX defined by :

$$g^t(v) = v_t$$

where $v_t(s) = v(s+t)$.

Observe that

$$\forall\, t \in \mathbb{R}\ \forall\, \gamma \in \Gamma\ \forall\, v \in SX \quad g^t(\gamma v) = \gamma g^t(v)$$

and this implies that the flow g^t induces another flow on the orbit space SX_Γ.

The `geodesic flow of` SX_Γ is the induced flow g_Γ defined by:

$$g_\Gamma(\Gamma v) := \Gamma g^t(v) = \Gamma v_t$$

It is immediate that g and g_Γ define two flows in the sense of definition 2.2.

Other two objects that will attract our interest later in the proof of proposition 3.16 are the `[strong] stable and [strong] unstable foliations of` g^t: these are some mainstream sets coming from hyperbolic dynamics. They are defined for each $v \in SX$ by

$$\mathcal{F}_s(v) := \{w \in SX \mid d_{SX}(g^t w, g^t v) \xrightarrow{t\to+\infty} 0\}$$
$$\mathcal{F}_u(v) := \{w \in SX \mid d_{SX}(g^{-t} w, g^{-t} v) \xrightarrow{t\to+\infty} 0\}$$

respectively.

This sets are intimately related with the horospheres, as it happens on the hyperbolic plane:

Proposition 2.4
Let $\xi \in \partial X$ and $x, y \in X$. Denote by v_x and v_y the geodesic rays such that $v_x, v_y \in \xi$ and

$v_x(0) = x,\ v_y(0) = y.$
Then,

$$\lim_{t\to+\infty} d_X(v_x(t), v_y(t)) = 0 \Leftrightarrow \beta_\xi(x,y) = 0 \Leftrightarrow \beta_\xi(o,x) = \beta_\xi(o,y)$$

Corollary 2.5

$$H_{x,\xi} = \{y \in X \mid lim_{t\to+\infty}(v_x(t), v_y(t)) = 0\}$$

Thus, the sets $\mathcal{F}_s(v)$ and $\mathcal{F}_u(v)$ can be rewritten as:

$$\mathcal{F}_s(v) := \{w \in SX \mid w(0) \in H_{v(0),v(+\infty)},\ w(+\infty) = v(+\infty)\}$$
$$\mathcal{F}_u(v) := \{w \in SX \mid w(0) \in H_{v(0),v(-\infty)},\ w(-\infty) = v(-\infty)\}$$

2.1.2 Hopf Coordinates

The aim of this subsection is to check that the space SX is topologically a product.

Notation 2.6
If $\triangle$ is the diagonal of ∂X, we write

$$\partial^2 X = \partial X \times \partial X - \triangle$$

Set an origin o of X. We define a map $\partial^2 X \times \mathbb{R} \to SX$ as follows: send every element $(\xi_-, \xi_+, s) \in \partial^2 X \times \mathbb{R}$ to the unique geodesic line $v \in SX$ (1.18) verifying

$$v(-\infty) = \xi_-,\ v(+\infty) = \xi_+,\ \beta_{\xi_+}(o, v(0)) = s$$

Notice that the third condition fixes the parametrisation of a geodesic line from ξ_- to ξ_+. This map is indeed an homeomorphism: it is bijective; continuous by the construction of the geodesic (1.18); and its inverse is continuous since Busemann functions are continuous on $\partial X \times X$.

Now we write our main three objects on this coordinates:

- The *geodesic flow* g^t is written as :

$$g^t(\xi_-, \xi_+, s) = (\xi_-, \xi_+, s + t)$$

- The *stable and unstable foliations* are written as :

$$\mathcal{F}_s(\xi_-, \xi_+, s) = (\partial X \times \{\xi_+\} - \triangle) \times \{s\}$$
$$\mathcal{F}_u(\xi_-, \xi_+, s) = (\{\xi_-\} \times \partial X - \triangle) \times \{s'\}$$

  where $s' = \beta_{\xi_-}(o, v(0))$.

- The *action* of Γ on SX is written as :

$$\gamma(\xi_-, \xi_+, s) = (\gamma\xi_-, \gamma\xi_+, s - \beta_{\xi_+}(o, \gamma^{-1}o))$$

To conclude, if we define the equivalence relation on $\partial^2 X \times \mathbb{R}$:

$$(\xi_-, \xi_+, s) \sim (\xi'_-, \xi'_+, s') :\Leftrightarrow \exists\, \gamma \in \Gamma \ \mid \ \xi'_- = \gamma\xi_-, \ \xi'_+ = \gamma\xi_+, \ s' = s - \beta_{\xi_+}(o, \gamma^{-1}o)$$

the third point above tells us that we obtain another homeomorphism:

$$\partial^2 X \times \mathbb{R}/\sim \;\longrightarrow SX_\Gamma$$

2.2 Patterson-Sullivan's Construction

Set X a proper, $CAT(-1)$ space, and Γ a group acting properly discontinuously on X.

2.2.1 Poincaré Series of a Group

The point of departure of this theory is the «orbital growth». First observe that, since the action $\Gamma \curvearrowright X$ is properly discontinuous, given a couple of points $x, y \in X$ we can count the number of points of the orbit Γy inside some compact neighbourhood of x [remark 1.22 (b)]. Thus, one could define :

$$\forall\, R > 0 \quad N_{x,y}(R) := \#\{\gamma \in \Gamma \mid d(x, \gamma y) \leq R\}$$

The `orbital growth function` is the power series

$$O_{x,y}(t) := \sum_{k=1}^{\infty} N_{x,y}(k)\, t^k$$

and similarly, the `spherical growth function` is the power series:

$$S_{x,y}(t) := \sum_{k=1}^{\infty} [N_{x,y}(k) - N_{x,y}(k-1)]\, t^k$$

which are related by the equality $S_{x,y}(t) = (1-t)\, O_{x,y}(t)$, telling us that they have both the same radius of convergence

$$\rho_{x,y} = \frac{1}{\limsup_{k\to\infty} [N_{x,y}(k)]^{\frac{1}{k}}} \in [0, \infty]$$

Hence, they are both convergent for $|t| < \rho_{x,y}$ and divergent for $|t| > \rho_{x,y}$.

Now we introduce a fundamental series in our discussion :

Definition 2.7
For every $x, y \in X$ the functional series

$$s \in \mathbb{R}_{>0} \mapsto P_{x,y}(s) := \sum_{\gamma \in \Gamma} e^{-s\, d(x,\gamma y)} \in]0,\infty]$$

is called the `Poincaré series of` Γ.

Set the quantity

$$\delta_{x,y} = log(\rho_{x,y}^{-1}) = \limsup_{R\to\infty} \frac{1}{R} log N_{x,y}(R) \in [0,\infty]$$

Lemma 2.8
If $s < \delta_{x,y}$ then $P_{x,y}(s) = \infty$ and if $s > \delta_{x,y}$ then $P_{x,y}(s) < \infty$.

Proof. The series $P_{x,y}(s)$ and $O(t)$ diverge or converge together. For, define the finite sets

$$J_k = \{\gamma \in \Gamma \mid d(x,\gamma y) \in]k-1,k]\} \quad \forall k \geq 1$$

and make the variable change $t = e^{-s}$. Observe that $\# J_k = N_{x,y}(k) - N_{x,y}(k-1)$ and this implies

$$\sum_{\gamma \in J_k} e^{-s\, d(x,\gamma y)} \leq [N_{x,y}(k) - N_{x,y}(k-1)]e^{-sk} \leq \sum_{\gamma \in J_{k+1}} e^{-s\, d(x,\gamma)}$$

Since

$$P_{x,y}(s) - \varepsilon = \sum_{k=1}^{\infty} \sum_{\gamma \in J_k} e^{-s\, d(x,\gamma y)}$$

for some constant $\varepsilon \in \mathbb{N}$ we have the result. ∎

Corollary 2.9
The quantity $\delta_{x,y}$ does not depend on the points x, y but only on Γ. Therefore, we can write

$$\delta_\Gamma = \delta_{x,y}$$

Proof. From the triangle inequality we obtain

$$d(y,\gamma y) - d(x,y) \leq d(x,\gamma y) \leq d(y,\gamma y) + d(x,y)$$

and this yields

$$e^{-s\, d(x,y)} P_{y,y}(s) \leq P_{x,y}(s) \leq e^{s\, d(x,y)} P_{y,y}(s)$$

so $P_{x,y}(s)$ has the same convergence type that $P_{y,y}(s)$, hence $\delta_{x,y} = \delta_{y,y}$. If now $x', y' \in X$ are other two points, some symmetry and transitivity imply that

$$\delta_{x,y} = \delta_{x',y'}$$

∎

Definition 2.10
The quantity δ_Γ is the `critical exponent of` Γ. The group Γ is said to be of `convergence type` if $P(\delta_\Gamma) < \infty$ and of `divergence type` otherwise.

Remark 2.11
One can find $\delta_\Gamma = \infty$ and this belongs to the divergence type case.

Example 2.12

(1) A divergence type group with finite critical exponent: any surface group (the fundamental group of a closed surface of genre ≥ 2) or convex-co-compact group.

(2) A divergence type group with infinite critical exponent. An example could be done from proposition 3.6 [CDST18] substituting the nilpotent group for a group with exponential growth such as the free group of rank 2 (suggestion from *R. Coulon*).

(3) A convergence type group. There is an example in [DOP00].

2.2.2 Conformal Densities

We construct a family of *finite* measures on ∂X and supported on $\Lambda(\Gamma)$.

Definition 2.13
A α-`conformal density` is a set $\mu = \{\mu_x\}_{x\in X}$ of finite measures on the limit set $\Lambda(\Gamma)$ such that

$$\forall x, y \in X, \quad \forall \xi \in \Lambda(\Gamma) \quad d\mu_x(\xi) = e^{-\alpha\beta_\xi(x,y)} d\mu_y(\xi)$$

The number $\alpha \geq 0$ is called the `conformal dimension`.
We say that μ is Γ-`invariant` if $d\gamma^*\mu_x(\xi) = d\mu_{\gamma^{-1}x}(\xi)$ $[= e^{-\alpha\beta_\xi(\gamma^{-1}x,x)} d\mu_x(\xi)]$ for every $x \in X$.

Remark 2.14
The notation $*$ in the statement makes reference to the image measure of μ_x by γ^{-1}.

Now we state the main result of this subsection:

Theorem 2.15 ([Sul79])
There exists a δ_Γ-conformal density which is Γ−invariant.

Remark 2.16
For the proof : a finite measure on a metric space is a Radon measure.

Proof. Set $o \in X$. Our prove assumes that Γ is of divergence type. A detailed proof of the convergence type case can be found on [Pat76]. Basically, it differs from this one in the correction of the convergence of $P_{x,o}(\delta_\Gamma)$ by means of the introduction of an auxiliary function $h(d(x,\gamma o))$ in its summand.

1^{st} step : Construct a finite measure on $\overline{X}$. Define, for $s > \delta_\Gamma$,

$$\mu_{x,s} = \frac{1}{P_{o,o}(s)} \sum_{\gamma\in\Gamma} e^{-s\, d(x,\gamma o)} D_{\gamma o}$$

where $D_{\gamma o}$ represents the Dirac measure centred in γo. The set $\{\mu_{x,s}\}_{s>\delta_\Gamma}$ is a family of probability measures on $\overline{X}$. Since $\overline{X}$ is compact then, the set $\mathcal{P}(\overline{X})$ of probability measures on $\overline{X}$ is a $weak^*$-compact subset of the Banach space of complex Borel regular measures on $\overline{X}$ endowed with the total variation norm $\mathcal{M}(\overline{X})$. This is a consequence of the (see the case of Riesz representation theorem on [Rud87],p. 129., and bear in mind that $C_0(\overline{X}) = C(\overline{X})$. Then apply Banach-Alaoglu). Thus, there exists $(s_n)_{n\in\mathbb{N}} \subset \mathbb{R}_{>\delta_\Gamma} : s_n \xrightarrow{n\to\infty} \delta_\Gamma^+$ such that μ_{x,s_n} converges to a Borel regular measure μ_x when $n \to \infty$ for some measure μ_x on $\overline{X}$.

2^{nd} step : supp $(\mu_x) \subset \Lambda(\Gamma)$. The partial sums of $P_{o,o}(s)$ are positive and strictly increasing. By monotone convergence,

$$\lim_{s_n\to\delta_\Gamma^+} P_{o,o}(s_n) = \infty$$

and we know that the action of Γ on X is properly discontinuous, so neighbourhoods of points of X only contain a finite number of points of the orbit Γo implying supp $(\mu_x) \subset \partial X$. Similarly, for points of $\partial X - \Lambda(\Gamma)$ there are neighbourhoods containing a finite number of points of $\overline{\Gamma o}$, by definition of limit set. Thus, supp $(\mu_x) \subset \Lambda(\Gamma)$.

3^{rd} step : Γ–invariance. We assume μ is δ_Γ-conformal (which is proved on step 4). Let $\phi \in \Gamma$ and $\xi \in \partial X$. By continuity of the Busemann cocycle, given $\varepsilon > 0$, there exist a δ-neighbourhood V_δ of ξ in $\overline{X}$ such that, for all $y \in V_\delta$:

$$|\beta_\xi(\gamma^{-1}x, x) - [d(\gamma y, x) - d(y,x)]\,| \leq |\beta_\xi(\gamma^{-1}x, x) - \beta_\xi(\gamma y, y)| \leq \varepsilon$$

Without lose of generality, assume that V_δ is a Borel set. We have

$$\begin{aligned}
\gamma^*\mu_{x,s}(1_{V_\delta}) := \mu_{x,s}(1_{V_\delta}\circ\gamma^{-1}) &= \frac{1}{P_{o,o}(s)}\sum_{\phi\in\Gamma} e^{-s\,d(x,\phi o)}1_{V_\delta}(\gamma^{-1}\phi o)\\
&= \frac{1}{P_{o,o}(s)}\sum_{\phi\,|\,\phi o\in\gamma V_\delta} e^{-s\,d(x,\phi o)}1_{V_\delta}(\gamma^{-1}\phi o)\\
&= \frac{1}{P_{o,o}(s)}\sum_{\phi\,|\,\phi o\in V_\delta} e^{-s\,d(x,\gamma\phi o)}1_{V_\delta}(\phi o)\\
&= \frac{1}{P_{o,o}(s)}\sum_{\phi\,|\,\phi o\in V_\delta} e^{-s\,d(x,\phi o)}1_{V_\delta}(\phi o)\cdot e^{-s\{d(\gamma\phi o,x)-d(\phi o,x)\}}
\end{aligned}$$

Now, since $\phi o\in V$, applying the continuity of the Busemann cocycle above yields

$$d(\gamma\phi o,x)-d(\phi o,x)=\beta_\xi(x,\gamma^{-1}x)+O(\varepsilon)$$

and therefore

$$\gamma^*\mu_{x,s}(1_{V_\delta})=e^{-s[\beta_\xi(\gamma^{-1}x,x)+O(\varepsilon)]}\mu_{x,s}(1_{V_\delta})$$

Thus, taking weak* limits when $s=s_n\to\delta_\Gamma^+$ and when $\varepsilon\to 0$ we obtain

$$d\gamma^*\mu_x(\xi)=e^{-\delta_\Gamma\beta_\xi(x,\gamma^{-1}x)}d\mu_x(\xi)=d\mu_{\gamma^{-1}x}(\xi)$$

4^{th} step: δ_Γ-conformity. The argument is the same as above. In this case one writes:

$$\mu_{x,s}(1_{V_\delta})=\frac{1}{P_{o,o}(s)}\sum_{\phi\,|\,\phi o\in V_\delta}\frac{e^{-s\,d(x,\gamma\phi o)}}{e^{-s\,d(x',\gamma\phi o)}}e^{-s\,d(x',\gamma\phi o)}1_{V_\delta}(\phi o)$$

■

Remark 2.17

The proof relies on just a weak limit and on the continuity of the Busemann cocycle.

2.3 Bowen-Margulis Measures

For a hyperbolic surface $\mathbb{H}^2_\Gamma$ there exists a natural «smooth» measure : the Liouville measure. For its construction, one has to define a density : this is a collection of measures $\{\mu_i\}_{i\in I}$ such that μ_i is defined on the open set $\phi_i(U_i)\subset R^n$ of a local chart (U_i,ϕ_i), where $\{(U_i,\phi_i)\}_{i\in I}$ defines an atlas; and such that each μ_i is absolutely continuous with respect to the Lebesgue measure in addition to the verification of some compatibility condition, as it is usual ([GHL04]). Finally, the Liouville theorem states that this measure is invariant under the geodesic flow g^t_Γ defined on $T^1\mathbb{H}^2_\Gamma$. The problem is that if Γ is non-elementary, finitely generated and $\mathbb{H}^2_\Gamma$ has infinite volume, then $\Lambda(\Gamma)$ is strictly contained in $\partial\mathbb{H}^2$ and the flow is no-longer ergodic for the

Liouville measure. A solution is to use a dictionary between the manifold and its boundary.

Definitions. A `metric measure space` is a measure space $(\Omega, \mathcal{B}, M)$ such that Ω is a metric space, $\mathcal{B}$ is the completion of the Borel σ-algebra of X (this avoids technical problems with null sets) and M is a (extended) Borel measure. We denote it by (Ω, M).

A `Radon measure` is a Borel regular measure finite on compact sets.

Remark 2.18

Borel finite measures on separable, complete metric spaces are regular ([Cou12], appendix C, proposition 34 and theorem 23). Therefore, they are Radon measures.

A `measure preserving system` will be a metric measure space (Ω, M) space together with a measure preserving transformation G, where G can be a flow φ^t or a group Γ acting properly discontinuously on Ω.

Dictionary. Set X a proper, $CAT(-1)$ space and Γ a non-elementary group acting properly discontinuously. We have the following correspondence :

$$\{g_\Gamma^t\text{-}\textit{invariant Radon measures on } SX_\Gamma\}$$
$$\updownarrow$$
$$\{\Gamma\text{-}\textit{invariant, } g^t\text{-}\textit{invariant Radon measures on } SX\}$$
$$\updownarrow$$
$$\{\Gamma\text{-}\textit{invariant Radon measures on } \partial^2 X\}$$

Hopf coordinates in section 2.1.2 makes this a trivial verification. One has to bear in mind that Lebesgue measure on $\mathbb{R}$ verifies a uniqueness property related to translation.

Bowen-Margulis measures. If μ is a Γ-invariant, α-conformal density, one defines the Radon measures:

- The `Bowen-Margulis current` M_{BM} on $\partial^2 X$:

$$dM_{BM}(\xi,\eta) = \frac{1}{[d_x(\xi,\eta)]^{2\alpha}} d\mu_x(\xi)\, d\mu_x(\eta)$$

  where $x \in X$ is a fixed point. Remark that, due to the α-conformity, this measure is independent of the choice of x.

- The `Bowen-Margulis measure` m_{BM} on SX :

$$dm_{BM}(\xi,\eta,t) = dM_{BM}(\xi,\eta)dt$$

The measure m_{BM} is obviously g_t-invariant due to the Hopf coordinates. It is also Γ-invariant (in the sense of measures, not densities) since for all $\gamma \in \Gamma$:

$$\left[\frac{d_x(\gamma\xi,\gamma\eta)}{d_x(\xi,\eta)}\right]^2 = e^{\beta_\xi(x,\gamma^{-1}x)+\beta_\eta(x,\gamma^{-1}x)}$$

(properties of Busemann cocycles) and μ is Γ-invariant. As a consequence, m_{BM} inducts a g^t_Γ-invariant measure on SX_Γ which be referred with the same name/notation.

We conclude that $(\partial X^2, M_{BM}, \Gamma)$ and $(SX_\Gamma, m_{BM}, g^t_\Gamma)$ are two well-defined measure preserving systems.

2.4 The Shadow Lemma

Once the existence theorem 2.15 was proved on the 70s the theory has been quickly developed where the cornerstone is Sullivan's shadow lemma below (not to be mistaken with the «shadowing» lemma from hyperbolic dynamics). It would be one of the main ingredients in chapter 3 (not for proposition 3.16, though).

Set X a proper, $CAT(-1)$ space and μ a Γ–invariant α-conformal density with Γ a non-elementary group.

Definition 2.19 (Shadow)
Let $x, y \in X$ and $r > 0$. The `shadow at infinity` of $B(y,r)$ seen from x is the set:

$$O_r(x,y) := \{\xi \in \partial X \mid [x\xi) \cap B(y,r) \neq \emptyset\}$$

Lemma 2.20

$$\forall\, \xi \in O_r(x,y) \quad d(x,y) - 2r < \beta_\xi(x,y) \leq d(x,y)$$

Proof. The second inequality is a property of the Busemann functions. For the first inequality observe that for $z \in [y\xi) \cap B(x,r)$ using the properties of Busemann functions:

$$\beta_\xi(x,y) = \beta_\xi(x,z) + \beta_\xi(z,y) = \beta_\xi(x,z) + d(z,y)$$
$$|\beta_\xi(x,z)| \leq d(x,z) \leq r$$

and therefore,

$$d(x,y) \leq d(x,z) + d(z,y) \leq r + \beta_\xi(x,y) - \beta_\xi(x,z) \leq \beta_\xi(x,y) + 2r$$

■

Lemma 2.21

$$\forall\, \varepsilon > 0\ \exists\, r_0 > 0 \quad | \quad \forall\, r \geq r_0\ \forall \gamma \in \Gamma, \quad \emptyset(\partial X - O_r(\gamma^{-1}x, x)) < \varepsilon$$

Proof. The goal is to estimate $\langle \xi, \eta \rangle_x$ for $\xi, \eta \in \partial X - O_r(\gamma^{-1}x, x)$ (recall the definition of d_x on section 1.2.3).
First of all, if $\xi \in \partial X - O_r(\gamma^{-1}x, x)$, then (extended remark 1.7 (2)) :

$$\langle \xi, \gamma^{-1}x \rangle_x = d(x, [(\gamma^{-1}x)\xi)) + O(\delta) \geq r + O(\delta)$$

Now let $\xi, \eta \in \partial X - O_r(\gamma^{-1}x, x)$. Then, by extended definition of the Gromov product:

$$\langle \xi, \eta \rangle_x \geq \min\{\langle \xi, \gamma^{-1} \rangle_x, \langle \gamma^{-1}x, \eta \rangle_x\} - \delta \geq r + O(\delta)$$

Thus,

$$d_x(\xi, \eta) = e^{-\langle \xi, \eta \rangle_x} \leq e^{-r + O(\delta)}$$

■

Corollary 2.22
For $r \gg 0$,

$$\mu_x(\partial X) - \max_{\xi \in \partial X} \mu_x(\{\xi\}) < \mu_x(O_r(\gamma^{-1}x, x)) \leq \mu_x(\partial X)$$

Proof. The first inequality is trivial. For the second inequality, observe that since Γ is not elementary, μ_x is not atomic and this implies :

$$\max_{\xi \in \partial X} \mu_x(\{\xi\}) < \mu_x(\partial X)$$

Let $m_0 \in (\max_{\xi \in \partial X} \mu_x(\{\xi\}), \mu_x(\partial X))$. If $A \subset \partial X$ is a Borel set, there exists $\varepsilon > 0$ such that if $\emptyset(E) < \varepsilon$ then $\mu_x(E) \leq m_0$ (otherwise there is a contradiction $\lightning$ with max; there is nothing exotic/*Radon* here, it is just immediate).
Now lemma 2.21 applied to $E = \partial X - O_r(\gamma^{-1}x, x)$ implies the result for $r \gg 0$. ■

Theorem 2.23 (Shadow Lemma)
If $x \in X$ then, $\forall\, r \gg 0\ \ \exists\, C > 0$ such that

$$\forall \gamma \in \Gamma \quad C^{-1} e^{-\alpha\, d(x, \gamma x)} \leq \mu_x(O_r(x, \gamma x)) \leq C e^{-\alpha\, d(x, \gamma x)}$$

Proof. Γ-invariance of the conformal density μ implies:

$$\mu_x(O_r(x, \gamma x)) = \mu_x(\gamma O_r(\gamma^{-1}x, x)) = \mu_{\gamma^{-1}x}(O_r(\gamma^{-1}x, x)) = \int_{O_r(\gamma^{-1}x, x)} e^{-\delta_\Gamma \beta_\xi(\gamma^{-1}x, x)}\, d\mu_x(\xi)$$

and we conclude using first lemma 2.20 for $y = \gamma^{-1}x$ after writing $-\beta_\xi(\gamma^{-1}x, x) =$

$\beta_\xi(x, \gamma^{-1}x)$; and then corollary 2.22:

$$e^{-\alpha d(x\gamma^{-1}x)}\mu_x(\mathcal{O}_r(\gamma^{-1}x, x)) \leq \mu_x(\mathcal{O}_r(x, \gamma x)) \leq \mu_x(\partial X)e^{-\alpha[d(x,\gamma^{-1}x)-2r]}$$

and

$$\mu_x(\mathcal{O}_r(\gamma^{-1}x, x)) \geq \frac{1}{2}[\mu_x(\partial X) - \max_{\xi\in\partial X}\mu_x(\{\xi\})]$$

for $r \gg 0$. ∎

Chapter 3

Theorem of Hopf-Tsuji-Sullivan

We assume all along this chapter that X is a proper, $CAT(-1)$ space and Γ a group acting properly discontinuously on X.

The objective is to provide satisfactory communication between the geodesic flow g^t_Γ, the action $\Gamma \curvearrowright \partial X$, the convergence/divergence type of Γ and the measure value of $\Lambda_c(\Gamma)$.

The main sources for this chapter come from some comments on the preliminaries of [Lin18] for section 4.1; and [Rob03], 1.E. for the remaining chapter, despite the fact that some details were more clear on [Sul79] and [Nic89].

3.1 Hopf's Descomposition

It is a must to introduce a characteristic of measure preserving systems before we state the main result. Check [Aar97], chapter 1 for more details.

Set (Ω, M, G) a measure preserving system such that Ω is locally compact, separable and M is σ-finite. Assume that G is a flow φ^t.

A point $x \in \Omega$ is said to be `positively recurrent` [`negatively recurrent`] if there exists a sequence $(t_n) \nearrow \infty$ of real numbers such that

$$\lim_{n\to\infty} \varphi^{t_n} x = x \quad \left[\lim_{n\to\infty} \varphi^{-t_n} x = x\right]$$

A point $x \in \Omega$ is said to be `positively divergent` [`negatively divergent`] if for every compact set $K \subset \Omega$ there exists a constant $T > 0$ such that $\forall\, t \geq T$

$$\varphi^t x \notin K \quad [\varphi^{-t} \notin K]$$

Assume that M is a Borel measure on Ω invariant by the flow φ. Then the Hopf decomposition theorem asserts that the space Ω decomposes into a disjoint union of two φ−invariant Borel sets, Ω_C and Ω_D, which satisfy the following properties:

(A) There does not exist a Borel subset $E \subset \Omega_C$ with $M(E) > 0$ and such that the sets $(\varphi^k(E))_{k\in\mathbb{Z}}$ are pairwise disjoint.

(B) There exists a Borel set $W \subset \Omega_D$ such that Ω_D is the disjoint union of sets $(W_k)_{k\in\mathbb{Z}}$ where each W_k is a translate of W under the flow φ^k.

According to Poincaré's recurrence theorem ([Aar97], 1.1.5) M−a.e. $\forall x \in \Omega_C$ is positively recurrent. On the other hand, by Hopf's divergence theorem, M−a.e. $\forall x \in \Omega_D$ is positively divergent. This implies that the sets Ω_C and Ω_D are unique up to sets of measure zero.

Definition 3.1
The measure-preserving system (Ω, φ^t, M) is said to be `conservative` if $M(\Omega_D) = 0$, and `dissipative` if $M(\Omega_C) = 0$.

Let's make some remarks.

Remark 3.2

(1) Notice that if the measure is finite then, due to (B) above, the system (Ω, φ, M) is conservative. In case you want to find a contradiction, write $W_k = \varphi^{t_k} W$ for some $k \in \mathbb{N}$, and suppose that this system is such that $M(\Omega) < \infty$ and not conservative. In this case, $M(\Omega_D) > 0$ and therefore

$$\infty > M(\Omega_D) = M\left(\bigsqcup_{k\in\mathbb{N}} W_k\right) = \sum_{k\in\mathbb{N}} M(W_k) = \sum_{k\in\mathbb{N}} M(\varphi^{t_k} W_k) = \sum_{k\in\mathbb{N}} M(W) = \infty$$

which is absurd because of the definition of W_k and because $M(W)$ must be > 0.

(2) The second observation is that the Hopf decomposition is the same for φ and φ^{-1} ((A) and (B) have indexation on $\mathbb{Z}$) we can say that M−a.e. $\forall x \in \Omega_C$ $[\Omega_D]$ is not only positively recurrent [divergent] but also negatively recurrent [divergent], again by Poincaré's recurrence theorem [Hopf's divergence theorem].

(3) The last remark is that if $f \in L^1(M), f > 0$ a.e. then, up to a set of measure zero, the conservative part Ω_C can be written as

$$\Omega_C = \left\{x \in \Omega \;\middle|\; \int_{\mathbb{R}} f(\varphi^t x)\, dt = \infty\right\}$$

See [Aar97], proposition 1.1.6 for the proof. The <u>assumption of separability</u> on Ω <u>is crucial</u>.

Definition 3.3
A Borel set $E \subset \Omega$ is said to be `wandering` if M-a.e. $\forall x \in E$:

$$\int_{\mathbb{R}} 1_E \circ \varphi^t(x)\, dt < \infty$$

Remark 3.4
The dissipative part Ω_D is maximal union of wandering sets.

Finally, we conclude with our definition of ergodicity:

Definition 3.5
The measure preserving system (Ω, φ^t, M) is said to be `ergodic` if every φ^t-invariant Borel set $E \subset \Omega$ (i.e. $\varphi^{-t}E = E$ a.e.) either satisfies $M(E) = 0$ or $M(\Omega - E) = 0$.

As a conclusion, if (Ω, φ^t, M) is ergodic, then it is either conservative of dissipative since Ω_C and Ω_D are φ-invariant Borel sets, so they are null or co-null set; and if $M(\Omega-\Omega_C), M(\Omega-\Omega_D) = 0$ there are points positively recurrent and divergent at the same time, which is a contradiction.

In particular, if (Ω, φ^t, M) falls in the ergodic, dissipative case, we have $M(\Omega_C) = 0$ and $M(\Omega - \Omega_D) = 0$ so M must be infinite after our first remark.

Case for countable groups. The same definitions apply to (Ω, M, G) when G is a countable group H acting on Ω. Essentially exchange the roles of the Lebesgue measure dt and the Haar measure dh of H; $\mathbb{R}$ and H; and γ and φ^t.

3.2 Tactic

Now we are in place to state the main result of this chapter:

Theorem 3.6
Let μ be a Γ-invariant, α-conformal density. Then, we have two complementary situations (A) and (B). In each situation, the statements from (i) to (iv) are equivalent:

(A) (i) $\sum_{\gamma\in\Gamma} e^{-\alpha\, d(o,\gamma o)} = \infty$
(ii) $\mu_x(\Lambda_c(\Gamma)) = 1$
(iii) $(SX_\Gamma, m_{BM}, g_\Gamma^t)$ is conservative and ergodic.
(iv) $(\partial X^2, M_{BM}, \Gamma)$ is conservative and ergodic.

(B) (i) $\sum_{\gamma\in\Gamma} e^{-\alpha\, d(o,\gamma o)} < \infty$
(ii) $\mu_x(\Lambda_c(\Gamma)) = 0$

(iii) $(SX_\Gamma, m_{BM}, g_\Gamma^t)$ is dissipative and non-ergodic.
(iv) $(\partial X^2, M_{BM}, \Gamma)$ is dissipative and non-ergodic.

We will prove it all along this chapter. Our plan for the proof of the equivalences in (A) is to «dismember» the theorem and then put the parts together again. (B) automatically follows if one rearranges this plan. We have labelled some propositions of next sections with letters :

$(i) \Leftrightarrow (ii)$	is (a) for $\Leftarrow$ and (b),(c) for $\Rightarrow$
$(ii) \Leftrightarrow (iii)$	is (d) and (e)
$(iii) \Rightarrow (iv)$	is (f) and (g)
$(iv) \Rightarrow (ii)$	is (h)

3.3 Part I: The Hardest

In this section we prove

$$(i) \sum_{\gamma\in\Gamma} e^{-\alpha\, d(o,\gamma o)} = \infty \Leftrightarrow \ (ii)\ \mu_x(\Lambda_c(\Gamma)) = 1$$

Proposition 3.7 (a)

$$\sum_{\gamma\in\Gamma} e^{-\alpha\, d(x,\gamma x)} < \infty \ \Rightarrow\ \mu_x(\Lambda_c(\Gamma)) = 0$$

Proof. If we denote $\Lambda(r) = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} O_r(x, \gamma_n x)$ and $(\gamma_n) = \Gamma$, observe that

$$\Lambda_c(\Gamma) = \bigcup_{r\in\mathbb{Q}, r\uparrow\infty} \Lambda(r)$$

Now, the result follows from Shadow lemma 2.23 and classical Borel-Cantelli lemma. ∎

3.3.1 A Borel-Cantelli Lemma

Here we state a generalised version of the so called Borel-Cantelli lemma. Recall that a corollary of this lemma (the original) is a special case of the $0-1$ law, which talks about the measure of independent events. And independent events are somehow chaotic events: one infers that an application of this lemma should imply something related to ergodicity/non-ergodicity. Furthermore the advantage of this lemma is that is almost elementary.

Lemma 3.8 ([Rob03], lemma 1)
Set M a finite measure over a space Ω and $\{A_t\}_{t\geq 0} \subset \mathcal{P}(\Omega)$ such that the function

$$\mathbb{R}_{\geq 0} \times \Omega \longrightarrow \mathbb{R}\,,\ (t,x) \mapsto 1_{A_t}(x)$$

is measurable. Assume that

$$\int_0^{+\infty} M(A_t)\,dt = \infty$$

and

$$\exists C > 0 \quad | \quad \forall T \gg 0 \quad \int_0^T \int_0^T M(A_t \cap A_s)\,dt ds \leq C \left(\int_0^T M(A_t)\,dt \right)^2$$

Then,

$$M\left(\left\{ x \in \Omega \,\middle|\, \int_0^{+\infty} 1_{A_t}(x)\,dt = \infty \right\} \right) \geq \frac{1}{C}$$

As a consequence one obtains :

Proposition 3.9 (b)

$$\sum_{\gamma \in \Gamma} e^{-\alpha\, d(x,\gamma x)} = \infty \Rightarrow \mu_x(\Lambda_c(\Gamma)) > 0$$

Proof. Logistical omission, see [Rob03] 1.E. ■

3.3.2 A Differentiation Theorem

The Lebesgue differentiation theorem is a classical result that establishes that every function $f \in L^1(\mathbb{R}^n, \mathcal{L}^n)$ has small variations locally on almost everywhere point $x \in \mathbb{R}^n$ and therefore it can be approximated by taken its mean value asymptotically. We generalise this result for $f \in L^1(\partial X, \mu_x)$ in the following lemma : the problem is that shadows are not necessarily balls in $CAT(-1)$. For $f \in L^1(\mu_x)$, we introduce its maximal function

$$f^*(\xi) := \lim_{t \to +\infty} \sup_{O \in \mathfrak{O}_r(t,\xi)} \frac{1}{\mu_x(O)} \int_O f\,d\mu_x$$

where

$$\mathfrak{O}_r(t,\xi) := \{ \mathcal{O}_r(x,\gamma x) \mid \gamma \in \Gamma,\ d(x,\gamma x) > t,\ \xi \in \mathcal{O}_r(x,\gamma x) \}$$

Remark 3.10
$\mathfrak{O}_r(t,\xi)$ is a countable set.

Lemma 3.11
Let $r \gg 0$. Then,

$$\mu_x - a.e \quad \forall \xi \in \Lambda(r) \quad f(\xi) = f^*(\xi)$$

where $\Lambda(r) = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \mathcal{O}_r(x,\gamma_n x)$ and $(\gamma_n) = \Gamma$.

Proof. The proof is almost the same as the classical Lebesgue differentiation theorem (see [Rud87], chapter 7). It changes in one part : we need to establish a kind of $5r$ lemma (see appendix, lemma A.8) for the shadows. Namely, if $G \subset \Gamma$, we prove that there exists some

$G^* \subset G$ such that the elements of $\{O_r(x,\gamma x)\}_{\gamma\in G^*}$ are pairwise disjoint and

$$\bigcup_{\gamma\in G} O_r(x,\gamma x) \subset \bigcup_{\gamma\in G^*} O_{5r}(x,\gamma x)$$

Denote $G = \{\gamma_i\}_{i\in\mathbb{N}}$ with decreasing distance $d(x,\gamma_i x) \leq d(x,\gamma_{i+1}x)$. Define by recurrence a subsequence of the indexation:

$$\begin{aligned} i_0 &:= 0 \\ i_{k+1} &:= \min\{i > i_k \mid O_r(x,\gamma_i x) \cap \cup_{j\leq k} O_r(x,\gamma_{i_j} x)\} \end{aligned}$$

If $i_k < i < i_{k+1}$ for some k, then $O_r(x,\gamma_i o) \cap O_r(x,\gamma_{i_j} x) \neq \emptyset$ for some $j \leq k$ and thus there is a geodesic ray from x crossing both $B(\gamma_{i_j} x, r)$ and $B(\gamma_i x, r)$. There are two cases depending on the order how the ray crosses the balls.

(a) The order of crossing is $\gamma_{i_j} \to \gamma_i$. Then, $O_r(x,\gamma_i x) \subset O_{3r}(x,\gamma_{i_j} x)$ after an argument with triangles.

(b) The order of crossing is $\gamma_i \to \gamma_{i_j}$. Then $B(\gamma_i x, r) \subset B(\gamma_{i_j} x, 5r)$ after the fact that $d(x,\gamma_i x) \leq d(x,\gamma_{i_j} x)$.

On both cases $O_r(x,\gamma_i x) \subset O_{5r}(x,\gamma_{i_j} x)$. If now we set $G^* = \{\gamma_{i_j}\}$ we have established the result. ∎

Proposition 3.12 (c)

$$\mu_x(\Lambda_c(\Gamma)) > 0 \Rightarrow \mu_x(\Lambda_c(\Gamma)) = 1$$

Proof. Suppose $\mu_x(\Lambda_c(\Gamma)) > 0$. We are going to prove something stronger than what is stated: if a Γ-invariant Borel set $A \subset \Lambda_c(\Gamma)$ has positive μ_x-measure, then $\mu_x(A) = 1$; i.e. $(\partial X, \Gamma, \mu_x)$ is an ergodic transformation. In this case, choosing $A = \Lambda_c(\Gamma)$ we will have the result.

Take such $A \subset \partial X$ and $r \gg 0$ verifying :

(a) $\mu_x(A \cap \Lambda(r)) > 0$ (which is possible because otherwise $\mu_x(\Lambda_c(\Gamma)) = \mu_x(\bigcup_{r\in\mathbb{Q}, r\uparrow\infty} \Lambda(r) = 0$ ↯).

(b) The conclusions of lemma 3.11 and Shadow lemma 2.23 are true.

Set $B = \partial X - A$ and apply lemma 3.11 choosing $f = 1_B$ and $\xi \in A \cap \Lambda(r)$ generic. The conclusion, is

$$\exists\, (\gamma_n)_{n\in\mathbb{N}} \subset \Gamma \quad | \quad d(x,\gamma_n x) \xrightarrow{n\to\infty} \infty \quad \& \lim_{n\to\infty} \frac{\mu_x(O_r(x,\gamma_n x) \cap B)}{\mu_x(O_r(x,\gamma_n x))} = 1_B(\xi) = 0$$

Now, apply lemma 1.2 and lemma 1.3 to the numerator and denominator, respectively, to obtain the proportionality (recall the notation)

$$\exists\, C > 0 \quad | \quad \frac{1}{C}\mu_{\gamma_n x}(O_r(x,\gamma_n) \cap B)) \leq \frac{\mu_x(O_r(x,\gamma_n x) \cap B)}{\mu_x(O_r(x,\gamma_n x))} \leq C\mu_{\gamma_n x}(O_r(x,\gamma_n) \cap B))$$

On the other hand, by Γ-invariance of B and by Γ−conformity of μ_x, we also have:

$$\mu_x(O_r(\gamma_n^{-1}x, x) \cap B) = \mu_{\gamma_n x}(O_r(x, \gamma_n) \cap B) \xrightarrow{n\to\infty} 0$$

Now, if we bear in mind the regularity of the Radon measure μ_x, the precedent translates into the construction of a sequence of open sets $(\Omega_k)_{k\in\mathbb{N}}$ on ∂X such that:
(1) $\lim_{k\to\infty} \emptyset(\partial X - \Omega_k) = 0$
(2) $\lim_{k\to\infty} \mu_x(\Omega_k \cap B) = 0$
(we need the open condition for the measure in the limit).
It seems that Hausdorff distance makes the space of compact subsets of ∂X a metric space. Again using regularity, up to extraction we can suppose that $\partial X - \Omega_k$ converges to a point $\zeta \in \partial X$; then

$$\mu_x(B - \{\zeta\}) = \mu_x(B - \lim_k [\partial X - \Omega_k]) = \lim_k \mu_x(B - [\partial X - \Omega_k]) = \lim_k \mu_x(\Omega_k \cap B) = 0$$

Finally, taking $\gamma \in \Gamma \mid \gamma\zeta \neq \zeta$ and using the Γ−invariance of B, we conclude $\mu_x(B) = 0$, as we wanted to show. ■

3.4 Part II: Hopf's Argument

In this section we prove :

$$(ii)\ \mu_x(\Lambda_c(\Gamma)) = 1 \iff (iii)\ (SX_\Gamma, m_{BM}, g_\Gamma^t) \text{ is conservative and ergodic}$$

Proposition 3.13 (d)
Let $v \in SX$. Then, $v(+\infty) \in \Lambda_c(\Gamma)$ if and only if there exists a compact set $K \subset SX$ such that

$$\int_0^{+\infty} 1_{\Gamma K} \circ g^t(v)\, dt = \infty$$

Therefore, $(SX_\Gamma, g_\Gamma^t, m_{BM})$ is conservative if and only if $\mu_x(\Lambda_c(\Gamma)) = 1$.

Proof. This one is easy. The first part is a consequence of the definition of radial limit set $\Lambda_c(\Gamma)$. For the second part, one applies the first one, the characterisation of conservativity from remark 3.2 and Hopf's coordinates. ■

The rest of the section is devoted to prove that the geodesic flow g_Γ is ergodic if it is conservative. The proof will be an adaption of Hopf's argument to an infinite measure environment. This argument was used by Hopf to prove that the geodesic flow g_Γ is ergodic when X is a closed surface exploiting the product structure of SX and *Fubini's* theorem. See [Cou12], chapter 4, it is very illustrative.

If the *Bowen-Margulis* measure m_{BM} is infinite, the set of bounded Lipschitz functions on SX_Γ is no longer dense in $L^1(SX_\Gamma, m_{BM})$ ([Cou12],appendix C, theorem 22 for the finite case), so one has to find a reasonable alternative to the Lipschitz condition:

Proposition 3.14
There exists a function $\Psi \in L^1(SX_\Gamma, m_{BM})$, $\Psi > 0$ with the following property :

$$\exists\, C > 0 \quad | \quad \forall\, v, w \in SX : d_{SX_\Gamma}(v,w) \leq 1, \ \frac{|\Psi(v) - \Psi(w)|}{\Psi(w)} < C d_{SX_\Gamma}(v,w)$$

Proof. Define the auxiliary function

$$\widetilde{h} : SX \to \mathbb{R},\ v \mapsto d_X(v(0), \Gamma o)$$

and the strictly positive function

$$\widetilde{\Psi} : SX \to \mathbb{R},\ v \mapsto e^{-4\delta_\Gamma \widetilde{h}(v)}$$

Ψ will be the function induced by $\widetilde{\Psi}$ on SX_Γ.
1^{st} step : $\Psi \in L^1(SX_\Gamma, m_{BM})$.
Let $v = (\xi\eta)$ be a geodesic line such that $v \cap B_X(o,r) \neq \emptyset$. Assume that $v(0) \in B_X(o,r)$. There are two observations:

- The geodesic segment $v \cap \overline{B}_X(o,r)$ has length $\leq 2r$.
- $\langle \xi, \eta \rangle_o := \frac{1}{2}[B_\xi(o, v(0)) + B_\eta(o, v(0))] \leq \frac{1}{2}[2\, d_X(o, v(0))] \leq r$. Thus,

$$\frac{1}{[d_o(\xi,\eta)]^{2\delta_\Gamma}} = \frac{1}{[e^{-\langle \xi,\eta\rangle_o}]^{2\delta_\Gamma}} < e^{2\delta_\Gamma r}$$

Bearing in mind the definition of the measure m and its invariance by the geodesic flow g^t we obtain:

$$m(SB(0,r)) < 2re^{2\delta r}$$

Now consider a fundamental domain $F \subset X$ for the action of $\Gamma \curvearrowright X$ centred in o. Then, we have $\widetilde{h}(v) = d_X(v(0), o)\ \forall\, v \in SF$ and

$$\int_{[SB(o,k+1)-SB(o,k)]\cap SF} \widetilde{\Psi}\, dm \leq \int_{[SB(o,k+1)-SB(o,k)]\cap SF} e^{-4\delta k}\, dm$$
$$\leq e^{-4\delta k} m(SB(o, k+1)) < 2ne^{-2\delta n}$$

To conclude the first step, we obtain

$$\int_{SX\cap SD} \widetilde{\Psi}\, dm = \sum_{k=0}^{\infty} \int_{[SB(o,n+1)-SB(o,n)]\cap SD} \widetilde{\Psi}\, dm < \sum_{k=0}^{\infty} 2ne^{-2\delta n} < \infty$$

so $\widetilde{\Psi} \in L^1(SX \cap SD, m)$, implying $\Psi \in L^1(SX_\Gamma, m_{BM})$.

2^{nd} step : Ψ verifies the requested property.
First of all notice that, by definition of $\widetilde{h}$, if $d_{SX}(v,w) < 1$ then $\sup_{t\in\mathbb{R}} d_X(v(t),w(t)) < 1$ and this implies

$$\frac{|\widetilde{h}(v)-\widetilde{h}(w)|}{d_{SX}(v,w)} = \frac{d_X(v(0),o)-d_X(w(0),o)}{d_{SX}(v,w)} \xrightarrow{d_{SX}(v,w)\to 0} 1$$

since

$$\inf_{t\in\mathbb{R}} d_X(v(t),w(t)) \leq d_{SX}(v,w) < 1$$

and

$$-[2\langle v(0),w(0)\rangle_o - d_X(v(0),w(0))] \leq d_X(v(0),o) - d_X(w(0),o) \leq [2\langle v(0),w(0)\rangle_o - d_X(v(0),w(0))]$$

Thus, using the mean value theorem for $\widetilde{\Psi}$, there exists $u \in SX$ with $u(0) \in (v(0),w(0))$ such that

$$|\widetilde{\Psi}(v)-\widetilde{\Psi}(w)| = 4\delta\widetilde{\Psi}(u)d_{SX}(v,w) = 4\delta e^{-4\delta(\widetilde{h}(u)-\widetilde{h}(w))}$$

and since $d_{SX}(v,w) < 1$:

$$\frac{|\widetilde{\Psi}(v)-\widetilde{\Psi}(w)|}{\widetilde{\Psi}(w)} = 4\delta e^{-4\delta(\widetilde{h}(u)-\widetilde{h}(w))}d_{SX}(v,w) \leq 4\delta e^{4\delta|\widetilde{h}(u)-\widetilde{h}(w)|}d_{SX}(v,w) < 4\delta e^{4\delta}d_{SX}(v,w)$$

Now taking $C = 4\delta e^{4\delta}$ this ends the proof. ■

This is and adaptation of Hopf's ergodic theorem for the flow case :

Theorem 3.15 (Hopf Ergodic Theorem, [RT18])
Set $(\Omega,\mathcal{B},M,\varphi^t)$ a conservative, measure preserving transformation with M σ-finite and Ω a topological space. Let $f,p \in L^1(\Omega,M)$ such that $p > 0$ M-a.e. Then,

(i) The following limits exist and are equal M–a.e. $\forall\, x \in \Omega$:

$$F^*(x) = \lim_{t\to+\infty} \frac{\int_0^t f\circ\varphi^s(x)\,ds}{\int_0^t p\circ\varphi^s(x)\,ds} \qquad F_*(x) = \lim_{t\to+\infty} \frac{\int_0^t f\circ\varphi^{-s}(x)\,ds}{\int_0^t p\circ\varphi^{-s}(x)\,ds}$$

(ii) The function F^* is finite M-a.e., T-invariant, and for any T-invariant subset $A \in \mathcal{B}$:

$$\int_A f\,dM = \int_A F^* p\,dM$$

(iii) Furthermore, the flow φ^t is ergodic if and only if F^* is constant M-a.e. for all $f \in L^1(\Omega,M)$. In this case,

$$F^* = \frac{\int_\Omega f\,dM}{\int_\Omega p\,dM} \quad M\text{-a.e.}$$

Now we prove the main result of this section :

Proposition 3.16 (e)

$$(SX_\Gamma, g_\Gamma^t, m_{BM}) \text{ conservative } \Rightarrow (SX_\Gamma, g_\Gamma^t, m_{BM}) \text{ ergodic}$$

Proof. Our goal is to prove that for all $\widetilde{f} \in L^1(SX, m)$, Γ-invariant the function $\widetilde{F}^*$ defined for all $v \in SX$

$$\widetilde{F}^*(v) = \lim_{t \to +\infty} \frac{\int_0^t f \circ g^s(v)\, ds}{\int_0^t \widetilde{\Psi} \circ g^s(v)\, ds}$$

is constant m-a.e., where $\widetilde{\Psi}$ is the function constructed on 3.14. The result will follow then due to Hopf's theorem 3.15 (iii), which uses conservativity ($\widetilde{F}_*$ will be the symmetric function of $\widetilde{F}^*$).

By a density argument which uses remark 3.15 (ii), it suffices to prove this only for all $\widetilde{f} \in C_c(SX)$, Γ-invariant.

Reductions made, the tactic is the following: we are going to exploit the product structure of the stable and unstable foliation as we have detailed on section 2.1.2. with *Hopf coordinates* and conclude using g_t-invariance.

Let $v \in SX$ and $w \in \mathcal{F}_s(v)$. We show that $F(v) = F(w)$. First of all, choose $a < 1$ and observe that due to the g^t-invariance, $\widetilde{F}^*(g^s) = \widetilde{F}^*(g^{s+a})$. Now let's create a monster: a straightforward computation show that the difference

$$\frac{\int_0^t \widetilde{f} \circ g^s(v)\, ds}{\int_0^t \widetilde{\Psi} \circ g^s(v)\, ds} - \frac{\int_0^t \widetilde{f} \circ g^{s+a}(w)\, ds}{\int_0^t \widetilde{\Psi} \circ g^{s+a}(w)\, ds}$$

is equal to the difference of

$$\frac{\int_0^t \frac{\widetilde{f}\circ g^s(v) - \widetilde{f}\circ g^{s+a}(w)}{\widetilde{\Psi}\circ g^s(v)} \cdot \widetilde{\Psi} \circ g^s(v)\, ds}{\int_0^t \widetilde{\Psi} \circ g^s(v)\, ds} \tag{3.1}$$

and

$$\frac{\int_0^t \widetilde{f} \circ g^{s+a}(w)\, ds}{\int_0^t \widetilde{\Psi} \circ g^{s+a}(w)\, ds} \cdot \frac{\int_0^t \frac{\widetilde{\Psi}\circ g^s(v) - \widetilde{\Psi}\circ g^{s+a}(w)}{\widetilde{\Psi}\circ g^s(v)} \cdot \widetilde{\Psi} \circ g^s(v)\, ds}{\int_0^t \widetilde{\Psi} \circ g^s(v)\, ds} \tag{3.2}$$

Our Promethean adventure now shows the following : since $\widetilde{f} \in C_c(SX)$ and $\widetilde{\Psi}$ has the property of proposition 3.14 (and verifies the remark 3.2 (2)) we see that expression in (3.1) goes to 0 as $t \to +\infty$ and the right hand factor in (3.2) does the same. The left factor in (3.2) remains bounded. Thus we have shown that, if $v = (\xi_-, \xi_+, s')$ in Hopf's coordinates,

$$\widetilde{F}^*(v) = \widetilde{F}^*(w) \quad \text{on} \quad (\partial X \times \{\xi_+\} - \triangle) \times \{s'\}$$

A symmetric argument shows that for $v \in \partial X$ and $w \in \mathcal{F}_u(v)$:

$$\widetilde{F}_*(v) = \widetilde{F}_*(w) \quad \text{on} \quad (\{\xi_-\} \times \partial X - \triangle) \times \{s''\}$$

Since $\widetilde{F}^* = \widetilde{F}_*$ m-a.e., applying now *Fubini's* theorem this implies that F^* is constant a.e. on $\partial X^2 \times \{s'\}$ (see [Cou12], lemma 4.1 in a finite measure context). This is a crucial step in the proof.
Finally, by g^t-invariance, we conclude that $\widetilde{F}^*$ is constant a.e. on $SX = \partial X^2 \times \mathbb{R}$, as we wanted to prove. ∎

3.5 Part III : The Rest

In this section we prove :

$$(iii)\ (SX_\Gamma, m_{BM}, g^t_\Gamma) \text{ is conservative, ergodic} \Rightarrow (iv)\ (\partial X^2, M_{BM}, \Gamma) \text{ is conservative, ergodic}$$

$$(iv)\ (\partial X^2, M_{BM}, \Gamma) \text{ is conservative and ergodic} \Rightarrow (ii)\ \mu_x(\Lambda_c(\Gamma))_x = 1$$

The following result is trivial after the dictionary of section 2.3 :

Proposition 3.17 (f)

$$(SX_\Gamma, m_{BM}, g^t_\Gamma) \text{ is ergodic} \Leftrightarrow (\partial X^2, M_{BM}, \Gamma) \text{ is ergodic}$$

Proposition 3.18 (g)
If $(\partial X^2, M_{BM}, \Gamma)$ is ergodic, then μ_x doesn't have any atom and $(\partial X^2, M_{BM}, \Gamma)$ is conservative.

Proof. Suppose for a contradiction that μ_x has some atom $\xi \in \partial X$ and $(\partial X^2, M_{BM}, \Gamma)$ is ergodic. Choose $\phi \in \Gamma$ such that $\phi\xi \neq \xi$ and consider the orbit $O = \Gamma(\xi, \phi\xi) \subset \partial X$. Observe that :
(1) $M_{BM}(\partial^2 X - O) = 0$ since the action is ergodic.
(2) The stabiliser $\Gamma_\xi = \{\gamma \in \Gamma \mid \gamma\xi = \xi\}$ is elementary, thus $\#\Lambda(\Gamma_\xi) < \infty$.
(3) $\Gamma_\xi \phi\xi$ can only accumulate on $\Lambda(\Gamma_\xi)$.
Thus,

$$\exists\, \gamma \in \Gamma \quad | \quad \gamma\xi \in \partial X - (\{\xi\} \cup \overline{\Gamma_\xi \phi\xi}) \neq \emptyset$$

Hence, $(\gamma, \gamma\xi) \in \partial^2 X - O$, ↯ with $M_{BM}(\partial^2 X - O) = 0$.
Finally, since Γ is countable and the action ergodic without any atom, it must be conservative.

∎

Proposition 3.19 (g)

$$\mu_x(\Lambda_c(\Gamma)) = 0 \Rightarrow (\partial X^2, M_{BM}, \Gamma) \text{ is dissipative}$$

Proof. We prove that $\partial X^2 - \Lambda_c(\Gamma)^2$ is contained in the dissipative part, i.e., $\partial X^2 - \Lambda_c(\Gamma)^2$ is reunion of wandering Borel sets. First of all, define the Borel function

$$\partial X^2 - \Lambda_c(\Gamma)^2 \to \mathbb{R}, (\xi, \eta) \mapsto D(\xi, \eta) = \min_{\gamma \in \Gamma} d(\gamma x, (\xi\eta))$$

and the subsets of Γ

$$\Gamma_{\xi,\eta} := \{\gamma \in \Gamma \mid d(\gamma x, (\xi\eta)) = D(\xi, \eta)\}$$

Therefore, for every $\mathcal{F} \subset \Gamma$ the set

$$E_{\mathcal{F}} := \{(\xi, \eta) \in \partial X^2 - \Lambda_c(\Gamma)^2 \mid \Gamma_{\xi,\eta} = \mathcal{F}\}$$

is a Borel set since we can write

$$E_{\mathcal{F}} = \left\{(\xi, \eta) \in \partial X^2 - \Lambda_c(\Gamma)^2 \;\middle|\; f_{\xi,\eta}^{-1}(D(\xi, \eta)) = \mathcal{F}\right\}$$

where

$$f_{\xi,\eta} \colon \Gamma \to \mathbb{R}, \gamma \mapsto d(\gamma x, (\xi\eta))$$

and we use the fact that ∂X is separable. Now one observes that:
(1) $\gamma E_{\mathcal{F}} \cap E_{\mathcal{F}} \Rightarrow \gamma F = \mathcal{F}$
(2) $\#Stab_\Gamma(\mathcal{F}) = \#\{\gamma \in \Gamma \mid \gamma\mathcal{F} = \mathcal{F}\} < \infty$ since $\mathcal{F}$ is finite.
Thus, the Borel set $E_{\mathcal{F}}$ is wandering, i.e., M_{BM}-a.e. $\forall v \in E_{\mathcal{F}}$

$$\int_\Gamma 1_{E_{\mathcal{F}}}(\gamma v) d\gamma < \infty$$

Finally, write

$$\partial X^2 - \Lambda_c(\Gamma)^2 = \bigcup_{\mathcal{F} \in \mathcal{P}(\Gamma)} E_{\mathcal{F}}$$

This implies that $\partial X^2 - \Lambda_c(\Gamma)^2$ is contained in the dissipative part. ∎

Appendix A

Connection with Hausdorff Dimension

The boundary at infinity ∂X of a proper, $CAT(-1)$ space is a separable (∂X is compact, section 1.2.1), metric (d_x, section 1.2.3) space. This condition permits to define a *Hausdorff measure* on ∂X. We conclude showing that if Γ acts properly discontinuously and co-compactly on X then, the Hausdorff dimension of $\Lambda(\Gamma)$ equals to the critical exponent δ_Γ and something else.

The main sources for this appendix are [PR18] for A.1 and [Bou95], [Hai10] for A.2.

A.1 Hausdorff Measures

See chapter 2.2 of [PR18].

It is well-known that the Euclidean space $\mathbb{R}^n$ comes naturally endowed with the Lebesgue measure, given for all Borel set $A \subset \mathbb{R}^n$ by

$$\mathcal{L}(A) = \inf \sum_i \varnothing(I_i)$$

where the infimum is taken among all the coverings of A by open rectangles I_i. We can generalise this idea to a more general context, namely, for *separable metric spaces* since it is the separability who permits the existence of countable coverings.

Set Ω a separable metric space.

Definition A.1

Let $A \in \mathcal{P}(\Omega)$. The s-`Hausdorff measure` of A is given by

$$\mathcal{H}^\alpha(A) := \lim_{\delta\to 0} \mathcal{H}^\alpha_\delta$$

where

$$\mathcal{H}^\alpha_\delta = \inf\left\{\sum_{i=1}^{\infty}[\varnothing(E_i)]^s \,\middle|\, A \subset \cup_{i=1}^{\infty} E_i,\ E_i \in \mathcal{P}(\Omega),\ \varnothing(E_i) \leq \delta\right\}$$

Remark A.2
This definition makes the function $\mathcal{H}^\alpha$ be a Borel measure on Ω ([PR18] theorem 2.7).

Proposition A.3 ([Hai10] lemma F.1)

(i) If $\mathcal{H}^\alpha(\Omega) < \infty$, then $\forall\, \alpha' > \alpha \;\; \mathcal{H}^{\alpha'}(\Omega) = 0$.

(ii) If $\mathcal{H}^\alpha(\Omega) = 0$, then $\forall\, \alpha' < \alpha \;\; \mathcal{H}^{\alpha'}(\Omega) = \infty$

It makes sense to define the following:

Definition A.4
The `Hausdorff dimension` of Ω, $dim(\Omega)$, is the number $\alpha \in [0,\infty]$ such that $\forall\, \alpha' > \alpha \;\; \mathcal{H}^{\alpha'}(\Omega) = 0$ and $\forall\, \alpha' < \alpha \;\; \mathcal{H}^{\alpha'}(\Omega) = \infty$.

Another important fact of Hausdorff measures is that we can estimate then by only using open balls:

Proposition A.5 ([Hai10] lemma F.2)
Let $A \in \mathcal{P}(\Omega)$ and consider the functions

$$\hat{\mathcal{H}}^\alpha(A) := \lim_{\delta\to 0} \hat{\mathcal{H}}^\alpha_\delta(A)$$

where

$$\hat{\mathcal{H}}^\alpha_\delta(A) = \inf\left\{\sum_{i=1}^{\infty} r_i^s \,\middle|\, A \subset \cup_{i=1}^{\infty} B_i,\ B_i = B(x_i, r_i),\ x_i \in A,\ r_i \leq \delta\right\}$$

Then,

$$2^{-s}\mathcal{H}^\alpha(A) \leq \hat{\mathcal{H}}^\alpha(A) \leq \mathcal{H}^\alpha(A)$$

A.2 The Connection

Set (Ω, d, M) a separable metric measure space (section 2.3) with M regular and X a $CAT(-1)$, proper space and Γ acting properly discontinuously and co-compactly on X.

Before arriving to the promised result which reflects the coincidence of $dim(\Lambda(\Gamma))$ and δ_Γ, we need to introduce a couple of things:

Definition A.6
We say that Ω is `Q-Ahlfors regular` $(Q > 0)$ if

$$\exists C > 0 \quad | \quad \forall x \in \Omega \, \forall r > 0 \quad C^{-1} r^Q \leq M(B(x,r)) \leq C r^Q$$

On the one hand, we have a corollary of the shadow lemma (theorem 2.23) :

Proposition A.7 ([Hai10], theorem 7.19)
If μ is a Γ-invariant α-conformal density, then, $\forall x \in X$ $(\partial X, d_x, \mu_x)$ is α-Ahlfors regular.

Proof. This requires first to prove that shadows $O_r(o, y)$ can be contained and contain certain balls on $(\partial X, d_o)$; this is a consequence of δ-hyperbolicity. Then, one exploits the co-compact condition to use the shadow lemma. ■

On the other hand, the following is a classical lemma (that comes form the Vitali covering theorem) applied on A.9 below :

If $B = \overline{B(x,r)}$ we denote $\forall s > 0, \; sB = \overline{B(x, sr)}$

Lemma A.8 (Covering Lemma «5r», [PR18] chapter 3.1)
Let $\mathcal{F}$ a family of non-degenerated closed balls of Ω such that

$$\sup_{B \in \mathcal{F}} \emptyset(B) < \infty$$

Then, there exists a sub-family $\mathcal{G} \subset \mathcal{F}$ countable at verifying :

(i) The balls $B \in \mathcal{G}$ are pairwise disjoint.

(ii) $\cup_{B \in \mathcal{F}} B \subset \cup_{B \in \mathcal{G}} 5B$

Proposition A.9 ([Hai10], lemma F.4)
If Ω is Q–Ahlfors regular then $dim(\Omega) = Q$ and

$$\forall A \in \mathcal{B}(\Omega) \;\; \exists C > 0 \quad | \quad \frac{1}{C} \mathcal{H}^Q(A) \leq \mu(A) \leq C \mathcal{H}^Q(A)$$

Finally, we get to the main result :

Theorem A.10 ([Sul79], theorem 8)
We have

$$dim(\Lambda(\Gamma)) = \delta_\Gamma$$

and

$$\mu_x \asymp \mathcal{H}^{\delta_\Gamma}$$

for any Γ-invariant α-conformal density μ.

Proof. The action $\Gamma \curvearrowright \Lambda(\Gamma)$ is ergodic (proof of proposition 3.12) and this implies that there is on $\Lambda(\Gamma)$ one and only one Γ-invariant α-conformal density of any dimension α (argument of [Rob03] corollary 1.8). Thus, $\alpha = \delta_\Gamma$ (theorem 2.15). Now, we obtain the desired result from propositions A.7 and A.9. ∎

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